FACTORS AFFECTING STUDENTS' PERCEPTIONS OF DIFFICULTY IN CALCULUS WORD PROBLEMS

Tracy CRAIG
University of Cape Town
Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, Cape Town, South Africa
email: tracy@maths.uct.ac.za

ABSTRACT

The purpose of the research project detailed in this paper was to ascertain, if possible, the answers to the research question that can be broken down into the following subquestions:

1) What factors contribute to students' perception of the difficulty level of a word problem?
2) How do students rank word problems in order of difficulty?
3) Are there differences between experts and novices in the ranking of problems?

The data collection instrument was designed in such a way that relative effects on difficulty level between characteristics could be determined, although no absolute effects, such as a quantitative measure of difficulty on an independent scale. For instance, it appeared that the context of the problem (concrete or abstract) had a greater effect on perceived difficulty level than the presence of a diagram. It was not possible to see, however, whether the difference was a subtle one or a clear and consequential one. The results of this study are informative, and it is the aim of this paper to summarise the central ideas on which the study was based, to outline briefly how the data was collected and to draw conclusions on the analysed data.

KEY WORDS/TERMS: mathematics education, word problems, hierarchy of difficulty, modal vectors

This paper outlines the motivation behind the study as well as several similar studies and their influence on this research project. Word problems are defined in terms of a problem classification framework, which is then used to draw up several problems for use in a survey of students and lecturers of mathematics at university level. Conclusions drawn from the data obtained from the survey are detailed as well as implications for further studies in this area.

Motivation behind the study

Students are known to find word problems difficult (Gerofsky, 1999; Craig & Winter, 1991/92 among others), yet experts in the form of mathematics lecturers or postgraduate students tend to find them easy, even mechanical (Schoenfeld, 1985; Larkin et al, 1980). As a mathematics educator, I hoped that this study would provide me with a new insight into why students find them difficult, and thus be able to teach them in a more accessible, less opaque, manner.

Research question and research design

The research question this project was designed to answer was essentially "What affects the perceived difficulty level of a word problem?" This question was subsequently expressed as three separate questions namely

1) What factors contribute to students' perception of the difficulty level of a word problem?
2) How do students rank word problems in order of difficulty?
3) Are there differences between experts and novices in the ranking of problems?

The approach taken to answering these questions was to run a survey in which first year university students, as well as lecturers, completed a questionnaire. This questionnaire consisted of five word problems that the person completing the questionnaire was required to rank in order of difficulty, without attempting to solve the problems first. The survey results were analysed to determine what characteristics of these word problems affected their relative difficulty.

Other studies have been carried out that compare the relative difficulties of word problem characteristics, such as context, arithmetic operations, readability, presence of diagrams, and whether the problems are algorithmic or interpretive. Several of these studies are outlined below along with their implications for the present study.

Similar studies and their relationships to this study

The problems in this study have been divided into the categories *algorithmic and interpretive*. Algorithmic problems are defined as problems that require the problem solver to carry out some calculation, the numerical solution to which is the aim of the problem. Interpretive problems, in contrast, require little or no calculation and require the problem solver to draw a conclusion drawn from some information given and his/her knowledge of mathematics. Galbraith & Haines (2000) made similar divisions in their study involving first year undergraduate students. Their problems are primarily involved with graphs and functions, such as factorising quadratics, or reflecting and translating a graph. The problems are divided into mechanical (equivalent to algorithmic), interpretive and constructive, where constructive can be understood to be a combination of the two former categories. The difficulty level of each problem was measured by the level of success that the students experienced when attempting to solve the problems. The results of Galbraith and

Haines (2000) show clearly that mechanical problems are easier than interpretive problems, which, in turn, are easier than constructive problems. A limited number of problems appeared on the questionnaire of this study, none of which were interpretive, due to constraints on the length of the questionnaire.

Caldwell & Goldin (1987, 1979) carried out a similar study at junior school level (1979) and secondary school level (1987). The problems that they presented to schoolchildren were all word problems categorised as *concrete or abstract*, and hypothetical or factual. Concrete and abstract problems are defined in terms of the realism of their context, that is concrete problems are set in a realistic context and abstract problems have no immediate real world analogy. Hypothetical and factual problems differ in that factual problems simply describe a situation, while hypothetical problems suggest a possible change in the situation. In the Caldwell & Goldin (1979 & 1987) studies, the difficulty level of a problem was measured by the number of students who successfully solved the problem. Caldwell & Goldin (ibid.) found that abstract problems were significantly more difficult than concrete problems, a finding which is reflected in this study.

Smith et al (1994) measured the *readability* of problems on a university statistics examination paper according to number of words, number of clauses, and two measures of lexical density. Lexical density is measured as the ratio of lexical words to grammatical words, either in total, or per clause. They accorded each problem a difficulty level by recording how many students successfully completed the problem. They found no correlation between the readability and difficulty level of the problems. The findings of this project are in agreement with those of Smith et al (1994). The readability of a word problem does not appear to affect the difficulty level, either perceived or actual.

Threadgill-Sowder & Sowder (1982) compared the difficulty level of problems presented in verbal format versus those presented with detailed *diagrams* and minimal wording. The difficulty level was measured by the number of students (in junior school) successfully carrying out the problem requirement. The results showed that students found the problems presented almost entirely in diagrammatic form significantly easier than those presented in verbal form only.

The studies listed above all required the students to carry out the problems and measured difficulty by the percentage of students solving them correctly. This study was intended to be rather different, in that the students were not required to complete the problems. Indeed, the students were given no opportunity to do so. Difficulty ranking was to be affected by their perceptions of the problems alone. The students were required to read the problems and rank them in order of the *perceived level of mathematical challenge* represented by each one. Individual students could therefore judge this difficulty level in different ways, such as number of variables, expected time required to solve the problem, the geometric shapes involved, etc. The students were free to decide for themselves which problems they expected to require the most cognitive effort to solve. The problems had to be chosen very carefully, therefore, according to strict criteria, to allow a comparison of which characteristics of the problems affected this perceived level of cognitive demand.

Defining word problems

Different theorists have defined “word problems” in various ways. Some mathematics educators define word problems by their structure, appearance and the inbuilt assumptions behind them (Verschaffel et al, 2000; Gerofsky, 1996; Pimm, 1995; Janvier, 1987; Lesh et al, 1987). Word problems have an easily recognisable structure and some assumptions are always made (by students and teachers), such as assuming that information not mentioned in the problem statements will not be required for successful problem-solving (Gerofsky, 1996). A definition of word problems by their use as a tool, rather than by their characteristics is often used (Boote, 1998; Schoenfeld, 1989). Word problems can be very useful as a means of illustrating practical uses of an algorithm, or as a modelling tool in physics or statistics. A third method for defining word problems is by creating a framework in which multiple types of mathematical problem can be placed, of which word problems are only one. Dowling (1998) constructs one such framework, and Craig & Winter (1990) construct another. It is the framework of Craig & Winter, strongly influenced by the three-level cognitive model of Kitchener (1983) that is the framework used in this research project. Kitchener's cognitive model suggests that real-life problems (ill-structured problems) cannot be modelled by school taught word problems (well-structured problems). If this model is correct, it calls into question the widely accepted belief (Verschaffel et al, 2000) that word problems are taught in order to teach techniques that can be applied to real-life problems. This belief is allied to the concept of transfer of technique from one sort of problem to another, a subject hotly debated (Evans, 1999; Lave, 1988; Walkerdine, 1988). Despite the arguments against word problems being included in the syllabus, however, they need to be defined if they are to be studied, and hence a problem classification framework was developed.

The problem classification framework

Using this framework (see Figure 1 below), word problems are defined as *disguised well-structured* problems, which can be divided and subdivided into various categories. *Algorithmic and interpretive* problems are defined where algorithmic problems require calculation, and interpretive problems require applying knowledge to interpret information practically. These categories were also used by Galbraith & Haines (2000). Both of these can be divided into *concrete and abstract* problems (although abstract interpretive word problems are rare). Concrete problems are set in a context that is non-mathematical and realistic, whereas abstract problems are set in a mathematical context with no immediate real world connection (see also Caldwell & Goldin, 1987, 1979). Finally, the problems can be categorised as having a single form of *representation*, or having multiple representations, such as diagrams and tables. Much work has been carried out on representations and the translations between them (Pimm, 1995; Wood, 1995; Buxkemper & Hartfiel, 1995; Lesh et al, 1987).

The problem typology and the data collection instrument

A list of problems was drawn up with at least one problem in each category defined by the framework. An example would be algorithmic – abstract – no diagram. In this way a typology of 14 problems was drawn up, a full description of which, with more detailed discussion, is available in Craig (2001). Five problems were selected from this typology to appear on a questionnaire, which was then distributed to students registered for one of three first calculus courses at the University of Cape Town. The problems were chosen carefully to be different, yet closely related enough that useful comparisons could be made. The characteristics of the problems chosen are illustrated in Table 2. An example of a problem (Problem D) on the questionnaire is

> A hollow cylindrical container has a circular base of radius 5 cm and vertical sides. A solid sphere of radius r is placed inside and at the bottom of the cylinder, and water is

poured in until the sphere is covered. What value of r will maximise the amount of water needed?

The collection of data

The survey was carried out at the University of Cape Town, South Africa. The students were all attending one of three first year calculus courses, designed for science and business science students, commerce students and engineering students. 660 responses were received from the students surveyed and 20 responses from the experts, who were postgraduates and lecturers in the Department of Mathematics and Applied Mathematics. The questionnaires were distributed during tutorial sessions and the students were allowed ten minutes during which to complete them. The tutors were carefully instructed in the requirements of the survey and did not allow the students to carry out the calculations, nor to discuss the problems amongst themselves.

Figure 1 Problem classification tree-diagram

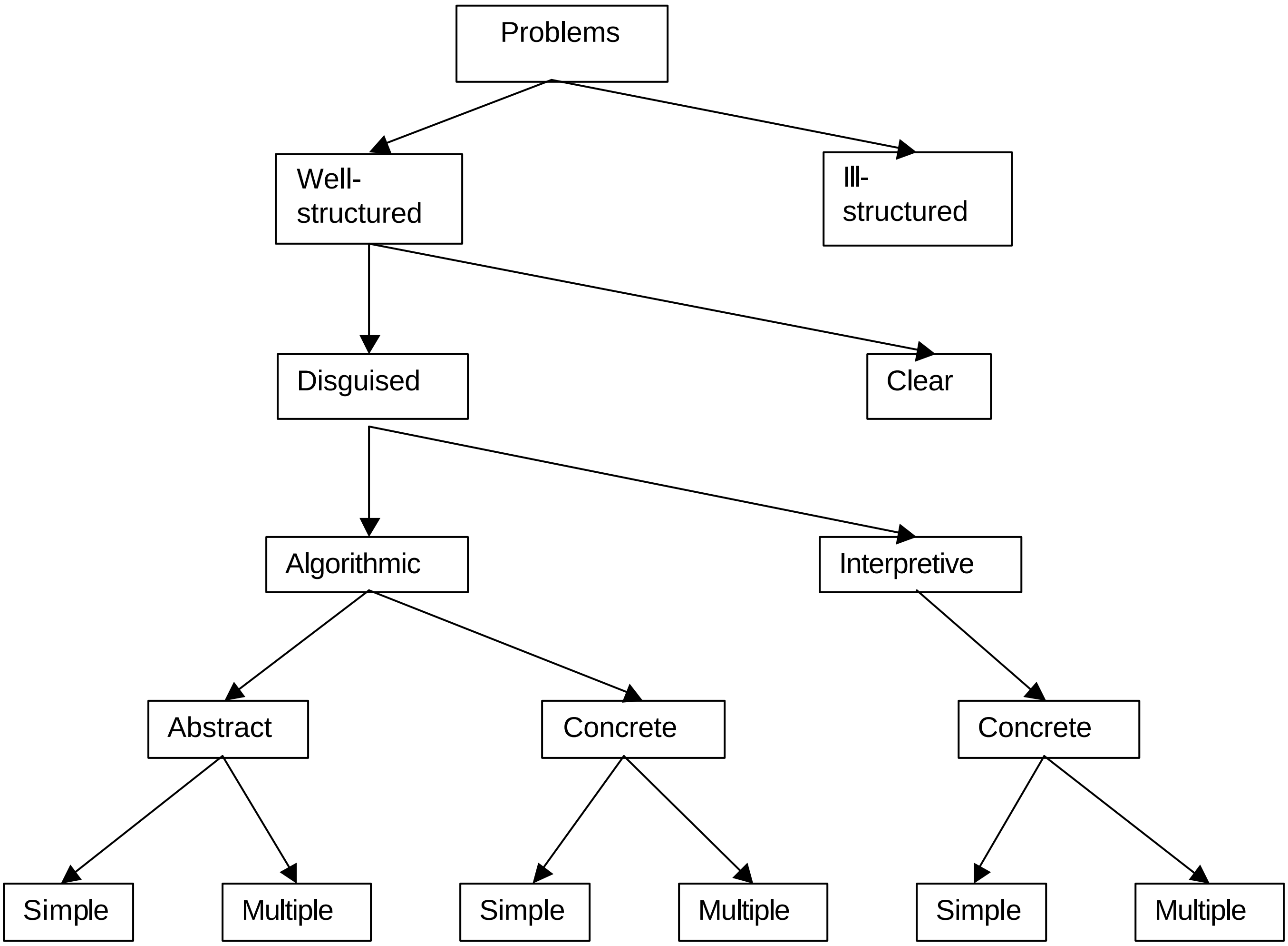

Table 2 Characteristics of word problems in questionnaire

	Context	Task requirement	Visual representation
A	Concrete	Algorithmic	No
B	Concrete	Algorithmic	Yes
C	Abstract	Algorithmic	Yes
D	Abstract	Algorithmic	No
E	Concrete	Algorithmic	Yes

The data

The information that the students had to complete on the questionnaire included a ranked list of the problems in the order easiest to most difficult, determined by their perception of the degree of mathematical challenge provided by each problem. This information was treated as a vector (for example BADEC) and a modal vector was calculated for each subpopulation, where the subpopulations were defined by gender, degree, or first language. The modal vectors were calculated in three ways, the most descriptive of which was the method by preference matrix (Siegel & Castellan, 1988), which measures the percentage of students ranking any one problem as preferred to any other (see Table 3 below). The modal vector was the same for every subpopulation except one that contained only 29 responses. This vector (AEBCD) ranked the problems in the following order (from easiest to most difficult):

Algorithmic - Concrete - No diagram (note: very common word problem)
Algorithmic - Concrete - Diagram (involving rectangles)
Algorithmic - Concrete - Diagram (involving circles)
Algorithmic - Abstract - Diagram
Algorithmic - Abstract - No diagram

Table 3 Absolute and relative preferences in the total population *

	A	B	C	D	E
A	•	620	629	614	542
B		•	401	448	264
C			•	427	191
D				•	142
E					•

	A	B	C	D	E
A	•	93.9	95.3	93.0	82.1
B		•	60.8	67.9	40.0
C			•	64.7	28.9
D				•	21.5
E					•

* Illustration by example of how to read the preference table: The first entry of 620 refers to the 620 students preferring problem A to problem B. The corresponding entry of 93.9 in the second table interprets the 620 students as 93.9% of the total population size.

Conclusions drawn from the data analysis

The ranking of the problems reveals preferences that are reflected in the studies mentioned earlier. A measurement was taken of the readability of each problem according to a lexical density test and the Flesch-Kincaid index. No correlation was found between readability and perceived difficulty, which correlates with the work of Smith et al (1994). Caldwell & Goldin (1987, 1979) observed that students find abstract problems harder than concrete problems, with which observation this study concurs.

A suggested hierarchy of difficulty, obtained from exhaustive analysis is:

- Familiar problems preferred to less familiar problems
- Concrete problems preferred to abstract problems (in agreement with Caldwell & Goldin, 1979, 1987)
- Problems with diagrams preferred to problems without diagrams (in agreement with Threadgill-Sowder & Sowder, 1982)
- Problems with rectangles preferred to problems with circles
- Readability indices (in agreement with Smith et al, 1994)

This hierarchy is apparent for every subpopulation within the novice population. The expert population tended towards this hierarchy as well, but not as clearly. The responses of the experts were widely spread with 14 different vectors from 20 questionnaires. The presence/lack of a diagram seemed to have less of an effect on expert perception of difficulty and unfamiliarity was less of a deterrent. The small number of experts taking part in the survey (20) did not allow for statistical analysis, but a descriptive analysis suggested that there was little correlation between the responses of experts and those of novices (students).

Contribution to the field and limitations of the study

The problem categorisation framework indicated in this paper is designed to apply to any mathematical problem, not solely word problems. The framework, along with the typology of problems developed with it in mind, can therefore be used to enable similar studies that can be considered as extensions of this one. For instance, a comparison can be made between the relative difficulties of clear and disguised problems. Another example would be to consider algorithmic and interpretive clear problems, rather than disguised problems. It is possible to extend the framework to include standard and non-standard problems (Craig & Winter, 1990; Yerushalmy & Gilead, 1999) that is, problems in their simplest form and ones that require simplification. This extension would be a difficult task, and one possibly open to debate.

In summary, the factors that appear to affect student perception of the difficulty level of a word problem are familiarity, context and visual representation in that order. Familiarity, particularly, plays a large role. Experts, in the form of mathematics lecturers and postgraduate students, do not have as clear a response. The varied responses from the experts suggests that there is no “correct” ranking of factors affecting difficulty, but that, as one gains in mathematical experience, one develops one’s own preferences for different types of problems.

Acknowledgements: The research detailed in this paper was conducted as a Masters in Mathematics Education dissertation under the supervision of Associate Professor Paula Ensor and Professor John Webb, both of the University of Cape Town. Their help is gratefully acknowledged, as is that of Dr Christien Thiart of the Department of Statistical Sciences, University of Cape Town. I would like to thank the Spencer Foundation and the University of Cape Town for their support in continued research.

REFERENCES

Boote, D. (1998) Physics word problems as exemplars for enculturation, *For the Learning of Mathematics* 18(2) pp28 – 33.

Buxkemper, A.C. & Hartfiel, D.J. (1995) Problem solving: discovery stage format, *International Journal of Mathematics Education in Science and Technology* 26(6) pp887-893.

Caldwell, J.H. & Goldin, G.A. (1979) Variables affecting word problem difficulty in elementary school mathematics, *Journal for Research in Mathematics Education* 10(5) pp 323-336.

Caldwell, J.H. & Goldin, G.A. (1987) Variables affecting word problem difficulty in secondary school mathematics, *Journal for Research in Mathematics Education* 18(3) pp187-196.

Craig, A. P. & Winter, P. A. (1990) An analysis of learners' engagement in mathematical tasks, *South African Journal of Higher Education* 4(1) pp59 – 68.

Craig, A.P. & Winter, P.A. (1991/92) The Cognitive and Epistemological Constraints on Praxis: The Learning – Teaching Dialectic in Mathematics, *Perspectives in Education* 13(1) pp45-67.

Craig, T. (2001) Factors affecting students' perception of difficulty in calculus word problems. MSc thesis, University of Cape Town.

Dowling, P. (1998) *The Sociology of Mathematics Education*. London: The Falmer Press.

Evans, J. (1999) Building bridges: Reflections on the problem of transfer of learning in mathematics, *Educational Studies in Mathematics* 39(1/3) pp23-44.

Galbraith, P. & Haines, C. (2000) Conceptual mis(understandings) of beginning undergraduates, *International Journal of Mathematical Education in Science and Technology* 31(5) pp651-678.

Gerofsky, S. (1996) A linguistic and narrative view of word problems in mathematics education, *For the Learning of Mathematics* 16(2) pp36 – 45.

Janvier, C. (1987) Translation Processes in Mathematics Education, in Janvier C. (ed.) *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Kitchener K.S. (1983) Cognition, metacognition and epistemic cognition, *Human Development* 26 pp222 – 232.

Larkin, J., McDermott, J., Simon, D.P. & Simon, H.A. (1980) Expert and Novice Performance in Solving Physics Problems, *Science* 28(20) pp1335-1342.

Lave, J. (1988) *Cognition in Practice*. Cambridge, Cambridge University Press.

Lesh, R., Post, T. & Behr, M. (1987) Representations and translations among representations in mathematics learning and problem solving, in Janvier C. (ed.) *Problems of Representation in the Teaching and Learning of Mathematics*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Pimm, D. (1995) *Symbols and Meanings in School Mathematics*. London: Routledge.

Schoenfeld, A. (1985) *Mathematical Problem Solving*. Orlando, Florida, USA. Academic Press, inc.

Schoenfeld, A. (1989) Problem solving in context, in Charles, R.I., Silver, E.A. (eds.) *The Teaching and Assessing of Mathematical Problem Solving*. Hillsdale, New Jersey. Lawrence Erlbaum Associates.

Siegel, S. & Castellan, N.J.Jnr (1988) *Nonparametric Statistics for the Behavioral Sciences*. McGraw-Hill.

Smith, N.F, Wood, L., Gillies, R.K. & Perrett, G. (1994) Analysis of Student Performance in Statistics, in Bell, G., Wright, B., Leeson, N. & Geake, J. (eds.) *Challenges in Mathematics Education: Constraints on Construction*. Mathematics Education Research Group of Australasia.

Threadgill-Sowder, J. & Sowder, L. (1982) Drawn versus verbal formats for mathematical story problems, *Journal for Research in Mathematics Education* 13(5) pp324-331.

Verschaffel, L., Greer, B. & de Corte, E. (2000) *Making Sense of Word Problems*. Lisse, The Netherlands. Swets & Zeitlinger.

Walkerdine, V. (1988) *The mastery of reason. Cognitive development and the production of rationality*. London. Routledge.

Wood, L.N., Smith, G.H. & Baynham, M. (1995) Communication Needs of Mathematicians, in Huntin, R.P., Fitzsimons, G.E., Clarkson, P.C. & Bishop, A.J. (eds.) *Regional Collaborations in Mathematics Education 1995*. ICMI.

Yerushalmy, M. & Gilead, S. (1999) Structures of constant rate word problems: A functional approach analysis, *Educational Studies in Mathematics* 39(1/3) pp185-203.