OF COURSE R^3 IS BLUE! DEVELOPING AN APPROACH TO TURN A MATHEMATICS COURSE INTO A MATHEMATICS EDUCATION COURSE[1]

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ABSTRACT

This study had as its objective to investigate the possibility of offering a mathematics-content course (in this case Linear Algebra) that is adequate to the professional development of mathematics educators (teachers, teacher educators and researchers); for reasons we will present and discuss, we started with the assumption that traditional content courses (in many cases the same as those presented to future mathematics researchers) were not adequate. The study consisted in the analysis of the transcriptions of videotaped lessons and other protocols collected at a four-months Linear Algebra course, taught to postgraduate students in a mathematics education postgraduate program in Brazil. We will focus on the presentation and discussion of the processes generated by the students' attempts to solve a mathematical problem, particularly on those relating to the production of meaning for the notion of space, and how the approach we took as professors (for instance, only to intervene to call their attention to recurrent statements or to divergences in the whole-group discussions) opened up a 'magic window' to the meanings they were producing for the notions involved, despite several Linear Algebra textbooks being available to them at all times. We will also argue that such a reading of the meaning production processes not only produces a very useful material for reflection during the course, but that it is in fact a necessary condition if we want content courses to be mathematics education courses that actually contribute to the professional development of our mathematics education students (teachers, teacher educators and researchers).

KEYWORDS: linear algebra, vector spaces, dimension, physical space, mathematics teacher education, meaning production

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Introduction

The education of secondary and high-school mathematics teachers follows, in Brazil as in many other countries, what we call the 3+1 format: the equivalent of three years of courses in mathematics, mostly the same courses a future researcher in mathematics would take, plus the equivalent of one year of courses in pedagogy related content. Many people in the mathematics education community have voiced their opinion that this is not adequate, but the issue has not yet been sufficiently studied (Wilson et al., 2001).

Since the second half of 1999 our research group has been working on developing an approach to the mathematical courses for future teachers that takes a view different of the 3+1

This paper reports an attempt to develop a *mathematics* course which would become, because of the way it was structured, a *mathematics education* course; one way to understand this is to think of this course as a service course directed to mathematics teacher education, as much as a Calculus course for engineers or computer scientists is not (or should not) be the same as that for the future mathematics researcher.

The course design

The core idea of the course was that while discussing mathematics (what we called 'the content') the students would also be discussing the processes they were going through (interaction, meaning production, obstacles/limits: 'the meta-content').

The subject chosen was Linear Algebra for postgraduate students (masters and PhD) at our postgraduate program. Many of them were at the time teaching mathematics at university level, for future teachers, some were school teachers and some were only working for their postgraduate degree; altogether 17 students. All sessions were videotaped.

We began the course proposing an investigation:

> " Let $\mathbf{R}^2 = \{ (a,b) ; a,b \in \mathbf{R}\}$
> Is it possible to exist a real vector space in which the vectors are the elements of $\mathbf{R}^2$, and such that its dimension is 3?"

One of the reasons for choosing this problem was that just to understand it one needs to take into consideration all the basic concepts of linear algebra, possibly leading, in our case to a revision of those concepts, that being part of their mathematical development. As we shall see, that did not happen so.

The students worked in groups of three and we said they could use any books they wanted to. One of the groups agreed to move to a separate room where their discussions were videotaped.

We told them that for the time being we would make no mathematical comments or answer mathematical questions to them; if they wanted to say something to us, we would listen, but they should not expect replies. Our silence was certainly responsible for the richness of the experience we were all going to go through. As the sessions went by the students were more and more open and making statements that could finally reveal to us what we were looking for: processes of meaning production for mathematics that were not only directed towards the mathematical meanings.

They had acess to all the information on the books, they had already taken Linear Algebra courses and many of them actually taught mathematics at university level, but almost all the discussion that followed would not be seen by a mathematician as 'mathematical'. That is when the title of this paper may make more sense: there was actually a discussion about whether or not, and why, $\mathbf{R}^3$ is blue.

The underlying theoretical framework

The crucial difficulty in a situation like this is how to understand the students' statements, in a very specific sense: how to categorise them without using primarily our own categories? Much more often than not, statements are interpreted according to categories which do not structure or organise the other person's thinking. A superb example of this is presented and discussed in G. Lakoff's book "Women, fire and dangerous things" (Lakoff, 1987).

The difficulty is precisely this: when a student makes a statement it is not sufficient to 'understand' the statement and to see whether or not we think it is a correct statement. It is also necessary to know what meaning *the student* produced for that statement (and the objects it refers to) and this is crucially related to why *the student* believes s/he can make that statement.

The support for this kind of reading in our work comes from the Theoretical Model of Semantic Fields (TMSF), developed by one of us as a tool to support teaching and research in mathematics education (Lins, 1992, 2001).

Its central notions are those of ‘knowledge’ and ‘meaning’. Knowledge is characterised as a statement a person makes and in which s/he believes (a statement-belief), together with a justification s/he has for making that statement. Meaning is characterised as what a person actually says about an object, in a given situation (activity); it is not everything s/he person could eventually say about that object. Meaning production and knowledge production always happen together, and objects are constituted through meaning production.

A third notion on the TMSF is relevant here, that of 'interlocutors'. It has to do with why a person thinks s/he can make a given statement in a given activity. We understand interlocutors as modes of meaning production that a person internalises as legitimate during his or her life; they are cognitive elements, not real people. In other words, in order to believe we *can* say something we must also believe that 'someone else' would say the same thing with the same justification. The notion of interlocutor is relevant because it allows us to explain why sometimes meaning production does not occur (typically expressed by statements like 'I did not understand' or by silences) or to explain divergent meanings being produced by two or more people. In this sense, communication is understood as the sharing of an interlocutor ('speaking in the same direction', rather than 'speaking in each other's direction').

Those three notions, knowledge, meaning and interlocutors, allowed us to keep the classroom activity as open as possible at the same time we could reliably trace the processes of meaning production in their dynamical character.

With the students

The typical first approach was to assume that "$\mathbf{R}^2$ *is* the plane" and to try to see *this* plane as a plane in *the* space ($\mathbf{R}^3$), so that the points in it would have three coordinates. Some students argued that this was fine, while others argued that the plane was still a plane and *its* dimension was still two. Although most of the groups browsed the textbooks available they did not effectively engage in finding the relevant definitions and discussing the problem in relation to them.

Quickly it became clear to us that for those students the initial question could only be asking for some kind of construction through which to place *the* plane on *the* space. From that moment we started to use (among us, who were conducting the course) the expressions 'natural plane' and 'natural space' to refer to the association between $\mathbf{R}^2$ and $\mathbf{R}^3$ to the cartesian, geometrical (perhaps physical, but we could not tell at that point), plane and space, respectively.

It also became clear that those natural spaces had intrinsically bound to them the natural operations of addition of vectors (points) and multiplication by a (real) number. Those students did not, at that point, produce meaning for $\mathbf{R}^2$ and $\mathbf{R}^3$ as (structureless) sets.

Besides the mathematical discussion, some important meta-questions were being asked: what had been done to the linear algebra they had studied in the past? Why was it that the books did not seem to be of any help?

The group that was in the separate room also tried this approach, but they went a bit further ahead with an interesting idea; working on the previous idea of immersing the plane into the space, combined with the idea that an answer depended on the operations, they tried to find an addition of vectors in $\mathbf{R}^2$ that would produce a vector in $\mathbf{R}^3$, something like,

$$(a,b) + (c,d) = (e, f, g)$$

That is when we want to come to the discussion of the colour of $\mathbf{R}^3$.

We were having a whole group discussion of their approaches. One of us (let’s call him ‘Green’) asked to speak. He said:

> [GREEN] I have always found the [sic] $\mathbf{R}^3$ much more interesting than the [sic] $\mathbf{R}^2$. I always see colour with the [sic] $\mathbf{R}^3$; for me it is blue. I was thinking of saying this. Perhaps because of the space?

He said it and left, excusing himself that he had some photocopying to do.

The first exchange we will look at happened immediately after Green left and has no visible relation to his statement. The second exchange happened at the end of that session, as Green asked to say a few more words.

At this point the students had already presented several ways of trying to make *the* space from planes, following their initial idea that what our original question actually asked was that.

> [DIVA] What is the plane for you?
>
> [ADES] Two linearly independent vectors. The union of the two linearly independent vectors.
>
> […]
>
> [ADES] I am understanding the plane as an infinite paper sheet.
>
> [DIVA] That infinite sheet of paper, it has how many dimensions? [sic]
>
> [ADES] Two. Only defined by the two LI vectors.
>
> […]
>
> [DIVA] But […] if it does not have the third dimension as you place one over the other it would make no difference. So it would have to have a minimum of thickness […] [2]

But Diva also admitted, when asked, that although she could not think of thickless planes inside the current activity, she would never mention thick planes to her students. How to make sense of this?

One interpretation is to say simply that Diva does not *really* know calculus or geometry or linear algebra. Our alternative interpretation is that Diva ‘has’ at least two ‘planes’: one is thickless

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[2] At another point it becomes clear that Diva is not just discussing Ades' idea: she actually cannot think of a thickless plane as she discusses our original problem.

and belongs to her Calculus lessons; the other is thick and belonged to our sessions. To be more precise, Diva had internalised (different) interlocutors and as she spoke in the direction of each of them it became legitimate to speak either of thick or of thickless planes.

We now move to the second exchange.

> [GREEN, near the end of a session] […] I would like to return to that question.
>
> [MILA] No, he said like, look guys, I think the $\mathbf{R}^3$ is blue. He speaks and leaves, he leaves things like that. […] it upset me a lot […] how can you see a colour like, blue? Why not yellow? Why not pink?

Mila took Green seriously and she was seriously questioning him; it is clear, by the tone of her voice, that she was actually disturbed by Green's statement. Mel said that "…he imagines the $\mathbf{R}^3$ blue, like", possibly because she believed that Green was not actually saying that the $\mathbf{R}^3$ *is* blue, but only that *he* imagines it so. There is a subtle but relevant difference between Mila and Mel. Mila believes Green's statement is about what $\mathbf{R}^3$ is and she cannot produce meaning for that (why *blue*?), while Mel is happy with Green imagining something about $\mathbf{R}^3$, having his own, idiosincratic, view, unrelated to mathematics. From a mathematical point of view, Mila leaves open the possibility that the mathematical object $\mathbf{R}^3$ have a colour, while Mel apparently does not, she seems to treat Green's statement as extra-mathematical. What could be the consequences of Mila's belief as she tries to understand and answer our initial question?

The exchange continues.

> [GREEN] Right, for me it is so natural that I can't even understand when you ask me.
>
> [BLUE] Because blue is a harmonious colour.
>
> [GREEN, talking to Maria Luiza] Do you also find [Mila's question] strange?
>
> [MARIA LUIZA] No, I was just about to ask what is the colour of the $\mathbf{R}^2$… [lots of laughs]
>
> [MILA] What about $\mathbf{R}$? [more laughs]
>
> [MUIARA] We associate the $\mathbf{R}^3$ to the space and the space for us, at school, at home, is the sky. It's just an association, nothing more.
>
> [BLUE] […] In a general way the $\mathbf{R}^3$ is this, it is the space. What's the space? When you look to the sky, the most beautiful thing to look at is the blue.
>
> [GREEN] Yeah, I had really thought of the space thing. But does it seem unreasonable to anyone [here]?

Blue seems to have a more aesthetical approach: the colour blue is harmonious and beautiful. Muiara seems to follow Mel's view. And Mila remains concerned with how to determine the colour of a space.

The crucial aspect, though, is that they all seem to be thinking, as they try to make sense of Green's statement, with a naturalisation of (the) $\mathbf{R}^3$ as the (physical) space, and the fact that we were supposed to be trying to solve a *mathematical* problem does not seem to disturb them; that suggests that for those students the mathematical $\mathbf{R}^3$ is really simply *the* space and that imposes to it all the natural properties of the physical space.

The first exchange, involving Diva and Ades had shown that also $\mathbf{R}^2$ was seen in a naturalised way (to the extent that Diva could not conceive a thickless plane), and the mathematical consequence is clear: from the outset the students *knew* that the answer to our problem should be 'no', because they *knew* that *the* dimension of $\mathbf{R}^2$ *is* 2, and the original problem could only produce a search for some tricky immersion of $\mathbf{R}^2$ into $\mathbf{R}^3$ or a search for an argument to show what they already knew.

Our silence allowed for all that: planes in the space to become three-dimesional, rotating planes to produce the space, displacing planes to produce the space, thick planes, the harmony and beauty of the sky. The interesting question is: had we not remained silent, had we directed the process of meaning production through intervention, through the correction of 'wrong ideas', where would those things be? Our answer is: in the students' street-smart backpack, that would have been left outside the classroom as they entered it for each session. And as they left the classroom they would leave the mathematical folder inside and take the backpack again, and hit the road (either to go home or to work). But some of the 'street' ideas might well remain in their pockets and who knows what the effect they will have on the possibility of them thinking mathematically.

We continue to follow the second exchange.

> [MEGA] You [Green] identify yourself with the blue, like thinking the $\mathbf{R}^3$ is blue. You think the $\mathbf{R}^3$ looks like blue. I'm not joking, I'm serious, for instance, I simpathyse with the number seven, I like the number seven, I think the five is too fat and the four has a big nose. […]
>
> [MUIARA] When I talk about space I find it difficult to understand it [in] $\mathbf{R}^1$ and $\mathbf{R}^2$ although when we are writing on a notebook, like that, you [leave, as as you write] a space […also] you are on a plane and [leave] a space [between paragraphs] So I have to make an effort to think of a space [sic] in $\mathbf{R}^2$. […] It is much more natural in dimension three for us to think of the space because there he occupies the whole volume.

Nosy numbers; space as in writing, as 'spacing' between words or paragraphs; a blue $\mathbf{R}^3$. This is the kind of material that made our course fruitful, through later discussions of what had been said by them, of why we remained silent and of what was requrired to understand the mathematical solution of the original problem.

The overall picture of the classroom is clear: a natural notion of space as something that is there to be occupied by something, a place where things (vectors, for instance, but also and equally, chairs and tables) can be, and a naturalised notion of $\mathbf{R}^3$ as *the* space; a natural notion of a plane as a smooth 'right' surface in which one draws and writes and rests objects, and a naturalised notion of $\mathbf{R}^2$ as *the* plane. The natural ones developd through the ordinary experience and other sign systems, the naturalised ones developed through experience in school (including university).

And what is wrong with that? For the vast majority of people we would say, nothing is wrong. For a few specialised professionals (some physicists working with superstrings, for instance) it would be a problem. But our main concern here is teacher education, so we must address the question from that point of view.

The evidence gathered in this course convinced us that through the discussion of their natural/naturalised notions of space and dimension, and by confronting those with the ones in linear algebra, we had achieved two objectives which are, from the point of view of mathematics teacher education, important. First, the students had the opportunity to reflect on their natural/naturalised ideas (which are almost always hidden in the background, and frequently conflicting with their mathematical 'correspondents') and on how these might affect their mathematical thinking, opening the door for real mathematical learning. Second through reflection on their own, lived, experiences, our students came to conceive the classrooms where they teach as places in which those processes are constantly happening, that is, as they face their own students they will be aware that similar processes might be happening even though with different objects as, for instance, when a school student encounters the notion of an infinite non-repeating decimal.

We think it was possible for our students to achieve both mathematical development (improved mathematical lucidity) and professional development (improved mathematical education lucidity).

Final remarks

Through a quite unexpected process the group arrived, reluctantly, at a solution: yes, it was possible to exist a real vector space in which the vectors are the elements of $\mathbf{R}^2$, and such that its dimension is 3; actually, infinitely as many (inducing the structure of the usual $\mathbf{R}^3$, using the fact that both $\mathbf{R}^2$ and $\mathbf{R}^3$ have the same cardinality). We discussed the basic notions involved (basis, dimension, the operations, inducing a structure using a bijetion) as well as how convinced they were of that (mathematically sound) answer.

Their reactions alone would generate a paper much longer than this one, but it is perhaps sufficient to say that one of the students, who had been very participative all along, said he was in shock, that he felt as if the ground had suddenly vanished from under his feet. His reaction was the starting point of a third phase of the course: "now you know that in mathematics there may be worlds quite different from our natural ones. So, keeping that in mind, let's look at some more of vector spaces". It had been *their* experience of surprise and 'shock', so they took it very seriously when we got to see families of paraboli associated to straight lines in the $\mathbf{R}^3$ with the usual structure.

All the time we made it very clear that our aim was not to correct their previous views, but to *add* a new possibility for meaning production and to help them to understand that sometimes one kind of meaning was more adequate, sometimes the other. And we stressed that everytime they had a student in front of them they should remember that maybe, just maybe, what was natural for the teacher was not natural at all for the student.

Evidence gathered in our other ongoing studies support the suggestion that what we met during this course is far from some kind of uncommited discourse (supposedly happening because we did not intervene to 'put an order' in it). Quite on the contrary, we think that discourse was committed and sound *in their terms*; we got a quite good understanding of what was supporting those students' thinking as they engaged with a *mathematical* problem.

A number of insights were gained during the course, and we will focus on three.

1) natural and naturalised objects have a high influence and a low visibility in students' thinking;

2) mathematics courses in the education of mathematics teachers should be designed as to create opportunities for those natural and naturalised objects to appear, and for two reasons:

   (i) to help students to understand that natural and naturalised objects are not, in most cases, what we are talking about, in mathematics, thus improving their chances of learning; and,

   (ii) to offer students the opportunity to discuss meaning production processes in a highly reflective way, as they were themselves the subjects of the processes being discussed.

3) it is possible that 'the mathematics of the mathematician', that is, mathematics as structured by the mathematician (calculus, analysis, linear algebra, algebra, and so on, and internally, inside each of those ‘blocks’) is not a suitable basis for the mathematical

education of mathematics teachers. Rather, we suggest that it is possible that the structures of natural and naturalised objects (space, measurement and quantity, for instance) might provide a more adequate basis.

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