ALGEBRA, COMPUTER ALGEBRA, AND MATHEMATICAL THINKING

Paul ZORN
St. Olaf College
1520 St. Olaf Avenue, Northfield MN 55057, USA
e-mail: zorn@stolaf.edu

ABSTRACT

Mathematical symbolism in general—and symbolic algebra in particular—is among mathematics' most powerful intellectual and practical tools. Knowing mathematics well enough to use it effectively requires a degree of comfort and ease with basic symbolics. Helping students acquire symbolic fluency and intuition has traditionally been an important, but often daunting, goal of mathematics education. Cheap, convenient, and widely available technologies can now handle a good share of the standard symbolic operations of undergraduate mathematics: differentiation, integration, solution of certain DEs, factoring and expansion in many forms, and so on. Does it follow that teaching these topics, and even some of the techniques, is now a waste of time?

The short answer is "no." On the contrary, as machines do more and more lower-level symbolic operations, higher-level thinking and deeper understanding of what is really happening become more, not less, important. Numerical computing has *not* made numerical viewpoints obsolete; neither will computer algebra render symbolic mathematics obsolete. The key question is how to help students develop that bred-in-the-bone "symbol sense" that all mathematicians seem to have. What really matters is that students use mathematical symbolism effectively to pose worthwhile problems in tractable forms. Once properly posed, such problems are well on the way to solution, often with the help of technology. The longer answer, explored in the paper, concerns choosing mathematical content and pedagogical strategies wisely in light of today's technology.

Introduction

What does it mean to know and do mathematics effectively at the tertiary level? How do the answers reflect the present and future, when mathematical technology, including symbol-manipulating technology, is already widely available, and will probably soon be ubiquitous?

What should college-level (tertiary) students in particular know and what should they be able to do, in order to be mathematically educated in a technology-rich environment? How can we teachers help bring students to this kind of knowing?

I approach these questions from a perspective that's fairly common in the United States: I'm a generalist mathematician who teaches reasonably pure mathematics to North American college students. About one-third of my students in an average class intend, with varying degrees of intellectual seriousness and interest, to complete a 4-year mathematics major. Only a small minority (10% or fewer) of students plan postgraduate study in mathematics. A more typical student plans to work after graduation in a technical but not university-level academic job, such as software engineering, database management, or high school teaching.

I am a practitioner of, not an expert researcher in, mathematics education, and so will not presume to offer advice on the education research agenda or how it should be carried out. What I hope to contribute is a teacherly and mathematical perspective on some content, techniques, and ideas related to symbolic mathematics that I think are mathematically important to today's tertiary students, and how I think students can be helped—sometimes with technological assistance—to acquire these advantages.

1 The technology background

Disputes over educational uses of mathematical technology have been around as long as the technology itself. Years ago one heard the "desert island" argument from opponents of instructional technology: Students who are permitted to use, say, calculators for school arithmetic might suffer disproportionately if later shipwrecked on low-tech islands. This argument is seldom heard anymore; it was killed either by the rising availability of cheap calculators or by the worldwide decline in passenger marine travel. In any event, there's no doubt that many students can now afford and keep readily to hand the technology needed to perform a huge share of the algorithms encountered even in tertiary mathematics. It's well known, for instance, that the TI-89 handles integrals, derivatives, partial fractions, and much more. But did you know that the TI-89 can also handle many of the residue calculations given as exercises in complex analysis texts? With powerful computer algebra systems such as *Maple* and *Mathematica* also becoming more affordable and available to students, the technology background has shifted markedly.

With the desert island argument no longer tenable, technology opponents resort to other arguments. Technology takes too much time to learn; students can't think in the presence of machines; technology use is just a post-modern cover for dumbing mathematics down—another nail in the coffin of civilization. I find these arguments unconvincing at best and dishonest at worst. How much do you think your students really struggle with technology as they pirate music files from the Internet? The dumbing-down argument is worst of all: it is simple "calumny" (as Tony Ralston observes in [**2**]) to equate technology-based reform with lowered intellectual standards or expectations.

This is not to deny, on the other hand, the existence of good, important, and (in my opinion) still open questions surrounding pedagogical uses of technology. Owning a calculator that "knows" how to expand rational functions in partial fractions does not necessarily obviate the need to understand something of the idea—and perhaps even of the process—by hand or by head.

At the school level, arguments over technology use often touch on the role and importance of paper and pencil arithmetic (PPA) in technology-rich environments. At one extreme are calculator abolitionists, asserting (with perhaps more vehemence than evidence) that calculator use is somehow inimical to reason—children, in this view, can either push buttons or think, but not both, and certainly not simultaneously. At the opposite end of the spectrum are other abolitionists, such as Tony Ralston, who advocate abolition not of calculators but of PPA itself, at least as an explicit goal of K-12 mathematics education. (One should hasten to add that Ralston also recommends greatly *increased* emphasis on mental arithmetic (and perhaps also on mental algebra) to replace PPA. His eloquent paper [**2**] is well worth reading.)

Beyond with this clash of opinions is, I believe, an important basic agreement on ultimate goals. In the end, most of us care far more about whether students can pose and solve novel and challenging problems than about what technology they may use along the way. What counts most is effective mathematical thinking, which comprises such elements as "symbol sense" and facility with mathematical structures; both are discussed in more detail below. What is mainly at issue, I believe, is whether technology can help, or must hurt, the cause of teaching students to think well mathematically.

2 Number sense and symbol sense

At the elementary level, what may matter less than PPA facility *number sense*, that intuition for numbers that includes such things as an ability to estimate magnitudes, an eye for obviously wrong answers, and an instinct for choosing (rather than necessarily performing) the arithmetic operation needed to solve a given problem.

At the secondary and tertiary levels, the mathematical symbols under study become much more general than numerals (which are, of course, symbols in their own right), and the degree of abstraction rises as students progress. The objects symbols stand for in more advanced mathematics might be unknown numerical quantities, functions, operators, spaces of various sorts, or even more abstract objects. At these higher levels of study the analogue of number sense is *symbol sense*, as defined by Arcavi [**1**] and others. Symbol sense is harder to define and delimit than number sense—appropriately enough, given the greater mathematical depth and breadth of, say, polynomial algebra as compared to integer arithmetic. (Arcavi lists at least seven aspects of symbol sense—only one of which involves actual symbolic manipulation.) Arcavi links symbol sense closely to *algebra*, asserting that acquisition of symbol sense is the proper goal of teaching algebra.

A student with good algebraic symbol sense should *see* that something is amiss with an "equation" like

$$(a + 2b)^4 = 17a^4 + 8a^3b + b^3a + \sqrt{ab}.$$

She should also *know*—without any calculation—that of

$$a^2 - b^2 = (a-b)(a+b) \qquad \text{and} \qquad a^2 + b^2 = (a+b)(a+b),$$

one is right and one is wrong. Similarly, one sees rather than computes that equations of the form

$$a^3 - b^3 = (a-b) \cdot (\text{something}) \qquad \text{and} \qquad a^3 + b^3 = (a+b) \cdot (\text{something})$$

can be arranged to hold, while

$$a^4 + b^4 = (a+b) \cdot (\text{something})$$

probably cannot.

In this paper I take broad views of both "symbol sense" and "algebra." By symbol sense I mean the general ability to extract mathematical meaning from and recognize structure in symbolic expressions, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover *new* mathematical meaning and structure. By "algebra" I mean symbolic operations in general, including not only algebra in the classical sense but also such things as formal differentiation and expansion in power series.

Definitions may differ, but *whatever* one means by "symbol sense", it's clear that tertiary-level mathematics takes a lot of it. Tertiary mathematics is a symbol-rich domain, and doing mathematics successfully at this level requires considerable comfort and sophistication with symbols. Above all, students need a clear sense of the things symbols represent, and how to extract meaning and structural information from symbolic expressions.

Perhaps this should all go without saying—who can doubt that symbols ought to mean something to students? In practice, however, we've all seen students floating untethered in the symbolic ether, blithely manipulating symbols but seldom touching any concrete mathematical ground. For example, many students struggle to make sense of a symbolic expression such as

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = \infty.$$

This is hardly surprising; after all, the statement's truth or falsity is far from obvious to a newcomer to infinite series. But a more basic source of difficulty, I believe, is that the expression's meaning—let alone its truth or falsity—is highly compressed in the symbolic representation. "Unpacking" the symbolism to reveal meaning and structure can be a daunting challenge in its own right, as we see often as our students confuse or conflate the terms and the partial sums of infinite series.

This brings me to my main questions:

1. How can we use technology—and symbol-manipulating technology in particular—to help students acquire symbol sense in the broad sense discussed above?

2. Where does better symbol sense lead? How can students use better symbol sense to understand mathematics more profoundly?

3 Building symbol sense

Technology can be used in many ways to help students make sense of symbols and symbolic expressions. We give two brief examples.

Example: Unpacking symbolic expressions

One approach to making sense of the densely packed symbolic expressions students encounter at the tertiary level is to use technology to "unpack" them and investigate their parts. (This is the essence of *analysis*.)

For the infinite harmonic series discussed above, for instance, the *Maple* command

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> s := n -> evalf ( sum(1/k, k=1..n)  ) :
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defines the partial sum function $s(n)$. Evaluating $s(n)$ is now easy for specific inputs n:

```
> s(10), s(20), s(30), s(40), s(50), s(60);

                  2.929, 3.598, 3.995, 4.279, 4.499, 4.680
```

The results show $s(n)$ increasing, although slowly, with n.

That's a good start, but it leaves open the deeper question of convergence or divergence. Further experimentation (and perhaps some hints) might eventually suggest successively *doubling* inputs to s:

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> s(10), s(20), s(40), s(80), s(160), s(320);

                  2.929, 3.598, 4.279, 4.965, 5.656, 6.347
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The situation is now much clearer; successive doubling of n causes essentially *linear* increase in $s(n)$ (by about 0.7 each time), and a useful analogy with logarithmic growth (which can lead to a rigorous proof of divergence) begins to appear.

Example: Looking closely at squares

Another technology-aided approach to giving meaning to symbols is to look very closely, from several viewpoints, at apparently familiar symbolic objects. Almost every American college student "knows," for instance, that

$$(x^2)' = 2x,$$

a fact that, while undeniably true, is almost entirely valueless without some deeper sense of what the symbolized objects and operations really mean. Here, too, students might use technology to help de-crypt the symbols, perhaps by plotting appropriate functions, zooming in on graphs, or calculating related derivatives.

For variety, let me suggest *another* approach to looking more "structurally" than usual at the squaring function, this time beginning from a numerical perspective.

What structure should a student see in the following list?

`1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 ...`

The first answer is obvious—even the dullest student with any recent memory of mental or paper-and-pencil arithmetic sees the squares of successive integers.

So far so good, but let's keep looking. Taking successive *differences* in the first list reveals the simpler pattern of successive odd numbers:

`1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 ...`

Taking differences *again* gives an even simpler list:

`2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 ...`

And so on. (Taking further differences soon loses its fascination.)

Starting from these basic structural ideas, students can move in many possible directions to explore—and perhaps solve—new but related structural questions:

- What happens if our original list arises by sampling not the basic quadratic function $f(n) = n^2$, but some *other* quadratic, say $g(n) = n^2 + 2n + 3$? Are the first differences *still* in arithmetic progression? Are the second differences still constant?

- How do differences behave if the original list samples the cubic function n^3? Or the exponential function 2^n?

- What happens if we move in the "opposite" direction, finding successive sums rather than differences? How does the "constant of summation" affect the results?

Quite different structural questions could also be explored. Students might notice, for example, that successive squares alternate between *exact* multiples of 4 and numbers of the form $4k + 1$. Or they might see pattern in the last decimal digits of successive squares:

`0 1 4 9 6 5 6 9 4 1 0 1 4 9 6 5 6 9 4 1 0 1 4 ...`

And so on, perhaps, into areas of modular arithmetic.

4 Beyond symbolics: exploring structures

We have argued that technology can help students build better symbol sense for tertiary mathematics. But why is symbol sense *worth* working to acquire? Where does it lead?

We should acknowledge first that, in actual practice and despite the presence of technology that could enable better things, a lot of tertiary mathematics still boils down to performing symbolic algorithms. As Ralston [**2**] says about college calculus in the USA:

> ... despite so-called calculus reform, the aim of most college calculus courses still seems to be to create a student-machine in which functions are fed to its maw and derivatives and integrals emerge at the other end.

In mathematical reality, of course, tertiary mathematics is about much more than algorithm performance, and technology may help us refocus attention where it belongs. The calculus, for instance, can be about mathematical objects and ideas—function, limit, derivative, differential equation, integral, infinite series—not just about formal calculations with these objects.

In my opinion, the true Holy Grail at the tertiary level is mathematical structure. Some italics may be in order:

> *Understanding basic mathematics profoundly means proficiency at detecting, recognizing, and exploiting structure, and at drawing useful connections among different structures.*

The preceding example illustrates most of these points: The basic structure of successive squares, once recognized and slightly manipulated, leads naturally to simpler or more complex structures, and to new, deeper, and more interesting questions.

There is nothing new about this focus on mathematical structure. Mathematics is frequently described, in one way or another, as the science of pattern. What may need emphasis, though, is the special importance of mathematical structure in *tertiary*-level mathematics. Here is where students meet new structures, and relations among them, in rich but potentially bewildering variety, ranging from abelian groups to planar graphs.

Quadratic polynomials: symbols reinforcing structure

We close with a final illustration of a pedagogical strategy—looking closely (perhaps using technology) at familiar objects—that focuses attention both on symbolics and on structures.

Quadratic polynomials are an excellent source of simple but not trivial examples; students should know them intimately and handle them often. The following example, although not particularly "technological", illustrates the value of studying familiar examples carefully, using symbolics, to reveal somewhat hidden structures.

(Before proceeding, we acknowledge in passing the good question of whether students should learn to manipulate quadratic polynomials *mentally*, as well as on paper and by machine. Ralston [**2**] recommends at least some mental manipulation. My hunch is that if quadratics are emphasized appropriately the question will become effectively moot: students will *automatically* acquire some mental facility with them. In any event, and whatever the medium of calculation or recording, students should *know*, not calculate, that $x^2 - 9$ factors as $(x-3)(x+3)$.)

In calculus, quadratic polynomials illustrate several important notions, including local linearity and "quadraticity", global nonlinearity, the meaning of the second derivatives, and geometric convexity. Quadratics also illustrate the possibility and the advantage of algebraic factoring, and more generally of the value of having convenient algebra formulas. One sees, easily, for instance, that the vertex of a quadratic polynomial lies midway between its roots, and that one root of a quadratic polynomial with rational coefficients is quadratic if and only if the other root is rational.

Example: Pythagorean triples and rational points

The rational roots property of quadratic polynomials has an interesting and perhaps unexpected "structural" consequence: there are infinitely many Pythagorean triples, and they correspond in a natural way to rational points on the unit circle.

The idea is as follows: Given a nontrivial Pythagorean triple (a, b, c) of integers, with $a^2 + b^2 = c^2$, we divide both sides by c^2. Renaming $x = a/c$ and $y = b/c$ gives a *rational* point (x, y) on the unit circle

$$x^2 + y^2 = 1.$$

Since the process can (essentially) be reversed, hunting for Pythagorean triples amounts to finding rational points on the unit circle. A few solutions are obvious; one is the "north pole" point, $(0, 1)$.

An ingenious way of finding other (indeed, essentially all) rational points is to find intersections of the unit circle with lines through $(0, 1)$ that have rational coefficients. Each such line that is not vertical has an equation of the form the line $y = mx + 1$, where the slope m is a rational number. Such a line intersects the unit circle at a simultaneous solution of

$$y = mx + 1 \qquad \text{and} \qquad x^2 + y^2 = 1.$$

A little algebraic work (by hand or by head) now produces the one-variable quadratic equation

$$x^2 + (mx + 1)^2 = 1.$$

This equation is easily solved for x. But we needn't bother, at least for the moment. Because all coefficients and the root $x = 0$ are all *rational* numbers, so is the other root. Because *every* line through $(0, 1)$ with rational slope cuts the unit circle in a rational point, we see that infinitely many rational points, and hence infinitely many Pythagorean triples exist. A little more work shows, moreover, that our recipe produces *all* rational points. Combining symbols, algebra, and various mathematical structures, we have solved a modest but nontrivial problem—and suggested methods of attack on many others. (Are there rational points on the circle $x^2 + y^2 = 3$? On $x^2 + 2y^2 = 3$)?) Somewhere, far in the distance, even the faint glow of elliptic curve theory can be detected.

5 Conclusion

As modern technology handles more and more of the algorithmic aspects of mathematics, even at the tertiary level, the importance of higher level mathematical thinking—symbol sense and facility with mathematical structure—become relatively *more* important. Used properly, high-level computing technology can help tertiary students see beyond the mechanics toward what matters most: mathematical structure.

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