GRAPHING CALCULATOR AS A TOOL FOR ENHANCING THE EFFICACY OF MATHEMATICS TEACHING

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ABSTRACT

New trends in mathematics teaching have emerged during recent years. These trends are connected with the current approach to school mathematics as a component of education in general, where mathematics is viewed as a tool for practical life. The rapid progress of technology is one of the aspects that have affected mathematics teaching at all levels, including the preparation of prospective teachers.

In the traditional teaching of mathematics the teacher passes complete information to the students and the students are passive recipients, while the integration of technology (computers, calculators, www resources) encourages and enables new approaches and procedures in mathematics teaching and learning - in particular a deeper investigation of problems, discovery of connections between phenomena etc.

Furthermore, technology can help to develop a better understanding of abstract mathematical concepts by their visualization or graphic representation; we can show relationships between objects and their properties. Such deeper understanding of concepts will in turn increase the ability of the students to acquire a better working knowledge of mathematics.

The article deals with the utilisation of graphing calculators in pre-service education of prospective mathematics teachers at the Faculty of Mathematics and Physics of Charles University in Prague. We use this type of technology in some of the subjects included in their study programs – specifically in “Didactics of mathematics” and “Methods of Problem Solving”. According to our experience it is advantageous to use graphing calculators in these subjects, especially when we introduce new important mathematical concepts, such as function, because most of the properties of functions can be found from the graphs drawn on the calculator display.

KEYWORDS: Preparation of teachers, graphing calculators, visualization, concept development, problem solving, equivalent and non-equivalent transformations, geometric transformations, derivatives.

1. Introduction

During several last years graphing calculators found their way gradually into secondary school mathematics of many countries in the world including the Czech Republic. According to the series of researches in this field [1], [3], [6] the using graphing calculators in mathematics teaching and learning can help the students to improve their knowledge and skills in some domains as concept development, problem solving, computation skills etc. Using graphing calculators in mathematics education bring also new methods of work - especially the possibility of *exploration and modelling of mathematical problems*, *multiple representation of mathematical problems* (numerical, algebraic, graphic, algorithmic representation) and *graphic support of the results obtained by algebraic procedures*. There are many positive aspects of the usage of graphic calculator in mathematics education (if we use this aid by proper way) and therefore it is necessary to react to this fact in pre-service education of mathematics teachers too.

Faculty of Mathematics and Physics of Charles University in Prague has amongst other programs also the one for prospective mathematics teachers, especially for the high schools. The study program for mathematics teachers is at the Master level, usually five years in duration. In the last two years of their study the prospective teachers get acquainted with the use of technology in mathematics teaching (including graphic calculators) in such subject as "Didactics of Mathematics" and "Methods of problem solving". Owing to our experiences (we investigated the influence of this tool on mathematics teaching and learning in classroom practice during 1993 - 1996) we have advised our students to use graphic calculators especially when teaching the topics of secondary school mathematics connected with the concept of function. In doing so, we emphasize to use it not only to study the definition of function, its properties and graphs, differential and integral calculus, limits of sequences, but also the solve equations, inequalities, their systems, investigating mutual positions of lines and regular conic sections and to study geometric transformations. In this respect it is appropriate to use graphing calculators in the *concept development process* (via its visualisation on the display), for the *simplification of the solution of mathematics tasks* and for *problem solving*.

According to psychologists, it is desirable to create an adequate image of concept issues out of its visualisation and also to involve students in a concrete experience with it. The process of concept acquiring is active and it consists of several parts: there is exerted visual cognition at first, followed by the verbal description of the image gained during the discussion with the teacher and classmates and resulting at the end by the image fixed through the students' own activity. The concept development is affected by many factors (e.g. student's motivation, knowledge, topic of learning), but teaching methods belong to the most important. In the particular parts of this procedure it is possible to use the graphing calculator as a tool for enhancing of the efficacy of this process.

In the following part the ideas mentioned above will be illustrated by several examples from different parts of school mathematics in such a way as we have used in didactical part of teachers' preparation at Faculty of Mathematics and Physics. In "Didactics of Mathematics" and "Methods of problem solving" we have used the graphing calculator TI-83 most of the time, because this type of calculator is used most often in our secondary schools. We have tested the algebraic calculator TI-89 in these subjects too.

2. Concept development in algebra, geometry and differential calculus

The solution of equations belongs among the basic skills in ***algebra*** course in the secondary school. The students have usually learnt to solve equations and inequalities using equivalent transformations. There is the basic concept - *an equivalent transformation* (i.e. the change of the algebraic form of the equation in such a way that the original, as well as the resulting equations have the same sets of solutions) and the students must understand the difference between the equivalent and non-equivalent transformation. The typical non-equivalent transformations are multiplication by an expression with a variable and also squaring. Traditional way for learning, which operations are equivalent, is memorising but using graphing calculator we can do it easier via graphical representation on the display. We can compare the solution sets at particular steps of solution process graphically.

Example 2.1

Where is a mistake?

$$x = 3$$
$$x(x-2) = 3(x-2)$$
$$x^2 - 2x = 3x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x-3).(x-2) = 0$$
$$x = 3 \ \vee \ x = 2$$

Solution

The solution set {2, 3} is wrong because the written procedure contains non-equivalent transformation which is applied from the first line to the second one – multiplication by (x – 2). We draw the graphs which are located at the second line and we can see (Figure 1) that there exist two common points, i.e. two solutions, in comparison with the first line $x = 3$. It means this transformation is not equivalent; more exactly, the multiplication by an expression is not equivalent if the expression is equal to zero.

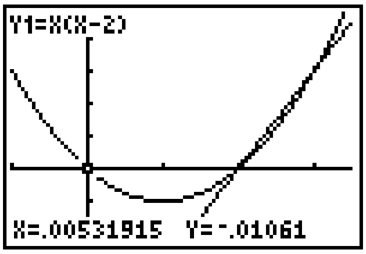

Figure 1

When solving equations or inequalities with radicals we usually use squaring and therefore we can obtain extraneous results that are not solutions of the original equation.

Example 2.2

Solve this equations $\sqrt{x+1} = x - 1$ for $x \in \mathrm{R}$.

Solution

At first we determine the condition for the radical ($x \geq -1$) to be defined and then we proceed by traditional method – we remove the radical by squaring:

$$\sqrt{x+1} = x - 1$$

$$x + 1 = (x - 1)^2$$
$$x + 1 = x^2 - 2x + 1$$
$$0 = x^2 - 3x$$
$$0 = x.(x - 3)$$
$$x = 0 \lor x = 3 \text{ (all solutions satisfy the condition for radical)}$$

When we substitute the results into the original equation we recognise that number 0 doesn't belong to the solution set. It is easy to show graphically that squaring transformation is not equivalent generally – we draw graphs from the first line (Figure 2) and from the second one (Figure 3) and we compare the number of common points.

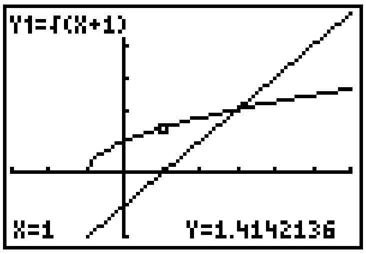

Figure 2

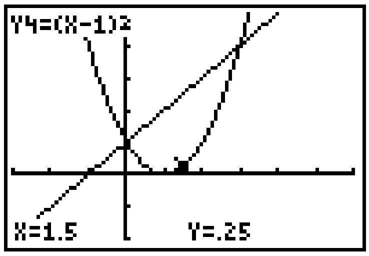

Figure 3

We can portray the concept of equivalent transformations by solving other examples while discussing whether or not squaring or multiplication is an equivalent transformation (e.g. find the solutions $\sqrt{x^2 + 2} = \sqrt{6 + 3x}$ or $\frac{6 - x}{x} = x + 4$). The graphing calculators let the students to concentrate on the introducing concept and not on the algebraic manipulation with expressions.

It is self-understood that we shall use graphing calculators in ***geometry*** for visualization of geometric objects in a plane (lines, circles and other regular conic sections) and investigate their mutual position and relationship. Furthermore, on the base of geometrical interpretation of algebra problems (if that is possible), we are able to develop the linkage between concepts in algebra and geometry. For example, when we solve the system of one linear and one quadratic equation we investigate the mutual position of line and regular conic section. It means, we can investigate the mutual position of two lines by comparing their slopes and y-intercepts, we can explore common points of conic section and line or the mutual position of two conic sections, depending on the type of equations (linear or quadratic) [7], [8]. This approach can assist in deeper conceptual comprehension that may influence the level of students' knowledge and skills in mathematics.

Another topic of geometry represents geometric transformations in a plane. The students learn to recognize distinct types of isometries (translations, axial symmetries, rotations) and also similarities (dilatation and shrinkage). The meaning of *isometry* (or the rigid motion) concept consists of the fact that isometry preserves distances and therefore it maps any geometric object to its image which is congruent with the original object. We can demonstrate it using graphic calculator effortlessly, because this tool (algebraic calculator) allow us to map any geometrical object to its image if we know the equations of the transformation.

Example 2.3

Find the image of the circle, the center of which has coordinates [-3, 3] and radius $r = 2$, in translation T: $(x, y) \to (x + 8, y)$. Compare the circle with its image.

Solution

We enter the parametric equations of the circle and equations of translation, we set up "thick curve" for the image and then we compare figures and radiuses (Figure 4, Figure 5) using the numeric functions of the calculator. Similarly we map lines, triangles etc. not only in translation but in other isometries too.

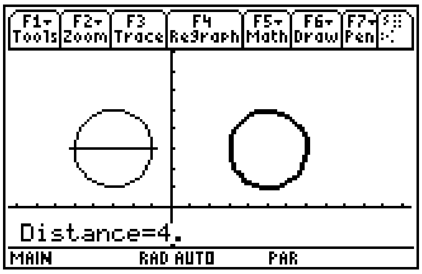

Figure 4

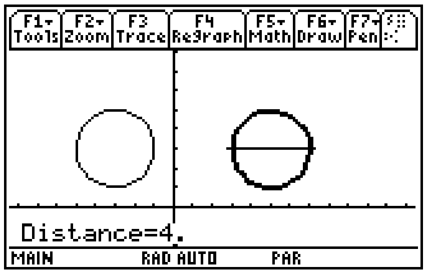

Figure 5

After that the students can confirm their observations about distances using the distance formula and coordinate geometry method.

The basic notion of ***differential calculus*** is the *derivative*. In mathematics teaching we usually utilize the graphical interpretation of the derivative to explain the students this important concept. Using graphing calculator it is easy to demonstrate this interpretation of the value of derivative at a given point as a slope of the tangent line to the graph of the function at the same point because the calculator is equipped by graphic command for drawing tangent line and expressing its equation (Figure 6).

In differential calculus the students learn to sketch the graphs of functions using the properties of the first and the second derivative, however, they sometimes forget that the first or the second derivatives are the functions too. We can draw the graph of function f together the graphs of its derivatives f', f''.

Example 2.4

Find the intervals in which the polynomial $f(x) = x^3 - 6x^2 + 9x$ is increasing or decreasing.

Solution

Let us write the function f and its derivatives in editor (Figure 7). At first we draw the graph of f and f'(thick line). To decide whether the function f is increasing or decreasing in some interval we have to determine where is the first derivative $f'(x)$ positive and where it is negative. It means to compute zeros of f'. We use the command "zero" and find the zeros $x = 1$ and $x = 3$ (Figure 8). The polynomial is increasing in $(-\infty, 1)$ and $(3, \infty)$, because there is $f'(x) > 0$; the function is decreasing in $(-1, 3)$ where $f'(x) < 0$. We can find the local maximum and minimum similarly. Using the graph of the second derivative of f we can find the intervals of concavity or convexity.

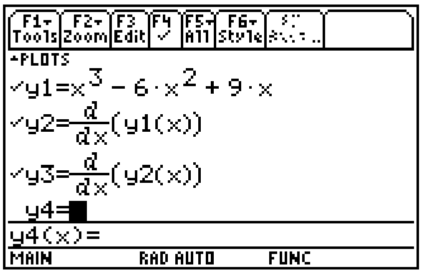

Figure 7

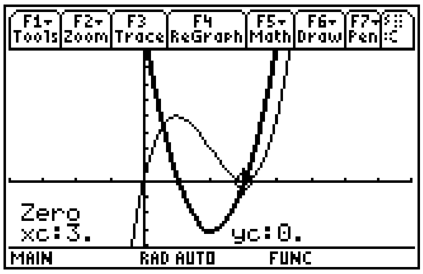

Figure 8

This procedure based on the graphic representation of abstract concepts is an invaluable instrument promoting the abstract idea to the knowledge and understanding of the topic.

3. Problem solving

The ability of using mathematics in practical life pertains to the important goals of mathematics teaching and learning. We have taught the students to solve the real world problems to show them the meaning of mathematics for their own life. However, this part belongs to the difficult ones. The solution process of the real world problems can be represented by the following schema [5]:

Real world		Model world	
Problem	® *translate*	**MP**	
		¯	*solve, calculate*
Solution	¬ *translate*	**MS**	

The first phase of the schema can be very laborious; transformation of the problem into the adequate mathematics model is the main phase in the process of the successful solution of the real problem. The students can use the graphing calculator not only in the second phase (calculation) but also in the third one. After finding the mathematics solution, the students need to verify that the results solve the real problem. The students are trained to recognise and interpret a "peculiar" solution (e.g. 2,51 pieces of bicycles or their negative number) but what about "nice" non-real solutions?

Example 3.1

In the warehouse the iron tubes are arranged into the layers in such a way that the tubes of higher layers fit in gaps of lower layers. We want to store 75 tubes into the layers with the lowest layer of 12 tubes. How many layers will we need?

Solution

We find the mathematics model - arithmetic sequence with the first term $a_1 = 12$, difference $d = -1$ and $s_n = 75$. We need to calculate the value of n, which means to solve the quadratic equation $n^2 - 25n + 150 = 0$. The roots of this equation are $n = 10$ and $n = 15$. Which of the results are solutions of real problem? We draw the graph of the arithmetic sequence with $a_n = 12 - (n - 1)$ on the display and we see that for $n = 15$ the number of tubes in 15-th line is negative -2 (Figure 9).

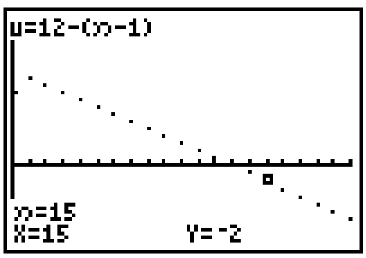

Figure 9

4. Conclusion

The procedures and examples mentioned in this contribution use the power of visualisation in learning process. The visualisation can help the student to understand and remember better the mathematical abstract concepts via their graphic representations, and graphic calculators can mediate this visualisation quickly and comfortably. Thus, graphing calculators represent the helpful tool for mathematics teaching and learning. However, the actual result depends on teachers themselves.

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