POLYNOMIALS IN THE CONTEXT OF LINEAR ALGEBRA: EXPRESSIONS? SEQUENCES? FUNCTIONS? VECTORS?

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ABSTRACT

Problem: Textbooks offer different definitions for polynomials. Examples:

- Expressions over a ring;
- Infinite sequences;
- Functions from the ring of coefficients into itself.

Mathematical and epistemological implications of the different interpretations will be discussed.
Methodology: In a one semester Linear Algebra course polynomials were defined as functions but the coefficient-criterion for equivalence was assumed. Vectors were defined as elements of a Vector Space (systemic definition). After the course the students were interviewed about polynomials and their role as vectors.

Findings:

- Two of the above interpretations of polynomials were present in the students' responses: Expressions and functions.
- Students evoked images that were never introduced in class, such as a curve for a polynomial, and a floating oriented segment for a vector.
- Students experienced difficulties in consolidating their contradicting prototypes of vectors and polynomials.
- The coefficient-criterion for polynomial-equality was rarely applied.
- Only one student used the systemic definition of a vector.

What is a polynomial? Textbooks often offer the following definitions:

Polynomials as expressions over a ring: In this interpretations polynomials are defined as expressions of the form $\mathbf{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}$ where $\mathbf{x}$ is a symbol which has no particular meaning and a_j are elements of the ring (Dubinsky & al., 1994). As such, two polynomials are considered equal if equal powers have equal coefficients.

Polynomials as infinite sequences with elements in a ring, of which all but a finite number equal zero. While in the previous definition $\mathbf{x^i}$ had *no particular meaning*, here it is defined as

$$\mathbf{x^i} \quad := \left\{ \mathbf{0}, \mathbf{0}, \ldots, \underset{\mathbf{i}}{\mathbf{1}}, \mathbf{0} \ldots \right\}$$

Together with the following definitions: $\alpha := \{ \alpha, 0, 0, \ldots \}$

$f+g := \{ \alpha_0+\beta_0, \alpha_1+\beta_1, \ldots \}$

and $f \bullet g := \{ \Sigma_{i+j=0}\alpha_i\beta_j, \Sigma_{i+j=1}\alpha_i\beta_j, \ldots \}$

We then have that $f = \{ \alpha_0, \alpha_1, \ldots \alpha_r, 0, 0, \ldots \}$ can be expressed uniquely in the form $f = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_r x^r$.

Here again two polynomials are defined to be equal if and only if $\alpha_0 = \beta_0, \alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots$ (Curtis, 1974). We call this definition of equality via the coefficients "the coefficient criterion for equality".

Polynomials as functions p(x) from the ring or field of their coefficients into itself. Consider the following definition:

> Let $f = \sum \alpha_i x^i$, where all but a finite number of α_i equal zero. In order that $f \in F[x]$, we define $f(\xi)$ as follows: Let $\xi \in F$. We define an element $f(\xi) \in F$ by $f(\xi) = \sum \alpha_i \xi^i$, and call $f(\xi)$ the *value of the polynomial f when ξ is substituted for x.*

(Compare, for example to Curtis, 1974, p. 168.)

The equality of polynomial functions is taken to be that of functions:

$f = g$ if and only if $f(\xi) = g(\xi)$ for all $\xi \in F$.

A **theorem** follows:

Two polynomial functions f(x) and g(x) are equal if and only if $\alpha_i = \beta_i$ for all i.

Proof:

$\alpha_i = \beta_i$ for all i $\Rightarrow f = g$ is obvious.

Not so obvious, although seldom treated with students, is the other direction:

$f = g \Rightarrow \alpha_i = \beta_i$ for all i.

It is easily proven in C[x] and F[x] , relying on the differentiability of polynomial functions over C and F: For any i , differentiate the equal polynomials i times, substitute in the ith derivative 0 for **x**, and you get $\alpha_i = \beta_i$.

What about other fields? In fact, the Coefficient Criterion for Equality does not hold for Polynomial Functions over any Zp with P prime. In Zp[x], x^P and x, polynomials of different coefficients, are equal functions. This follows from **Fermat's Little Theorem**:

Let p be a prime which does not divide the integer a, then a(p-1) = 1 (mod p).

Sometimes Fermat's Little Theorem is presented in the following form:

Corollary:

Let p be a prime and a any integer, then $a^p = a \pmod p$.

Research literature

Rauff (1994) deals with students' difficulties when operating on polynomials as expressions, without referring to them as functions. Harel (2000) claims that difficulties with vector spaces of functions in general (and of polynomials in particular) arise from the fact that the students have not formed the concept of a function as a mathematical object. To use his words: "as entities which they can treat as inputs for other operations" (there, p. 181). Using APOS terminology (Asiala & al., 1996) one might describe the need for the concept of function to have developed from action via process into object, in order for the student to be able to treat polynomials as members of a vector-space, and hence as vectors (The systemic definition of vectors, Syrpinska, 2000).

Dorier & al.(2000) treated vector spaces of polynomial functions and examined students' operating with specific values of the functions and their derivatives. So did Rogalsky (2000).

I did not find research on the flexibility required of students for shifting from one definition (interpretation) of the concept to the other, or the intuition students might or might not have regarding the coefficient criterion for equality.

Methodology of the reported research. Fifteen students took a one-semester course in linear algebra at a college for prospective high-school teachers of technological subjects. This was a first semester in the first year of their college training with no preparatory course in mathematics.

Teaching of this course tried to follow principles derived from the theoretical perspective APOS (Asiala et. Al 1996). Those teaching according to this perspective often use the ISETL software in undergraduate mathematics courses, but due to technical problems ISETL was used only partially in this course – only at its opening phase. Some ISETL activities were dedicated to the amelioration of the concept *function* in the minds of the students, bringing it closer to the level of object. It is accepted by APOS-oriented researchers that a significant development of the concept of function is a pre-requisite to the student's ability to construct adequate linear algebra concepts. For example: The construction of linear-combinations as functions with input scalars and vectors and output a single (new) vector ([1]). Similarly, the ability to treat polynomial functions as objects, to operate upon them the vector-space operations, and consider them members of a vector-space, also depends upon the student's previous development of a polynomial function as object.

A general characteristic of ISETL is that some of the more effective activities it enables can only be carried out on finite sets. Hence using ISETL in a linear algebra course naturally deals with finite fields Zp and vector-spaces over them. Hence a distinction between the different interpretations of polynomials arises in such a course.

<u>Polynomials and vectors in the course</u>

- The term Vector was first introduced with tuples, then broadened to other examples, and finally to the general (systemic) definition: A vector is an element of a vector space (See Fischbein, 1995, Sierpinska, 2000, on systemic thinking).
- Polynomials were defined as functions, and dealt with over R only.
- Vector spaces of polynomials over R were dealt with throughout the course;
- The coefficient-criterion for equivalence was presented and taken for granted (no proof).

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[1]) RUMEC - Research in Undergraduate Mathematics Education Community (2001). *Initial genetic decompositions for topics in linear algebra.* Unpublished report.

The interviews

Of the 15 students, 12 agreed to be interviewed after the course. The interviews consisted of a structured questionnaire, were conducted individually, with each interview lasting about 45 minutes, and were video-recorded. Question no. 8 was constructed to examine the concept of polynomial:

Question no. 8:

One. What is this: $x^4 + x^3 + 7$.

Students who did not identify this expression as a polynomial were reminded of this term. If the question "what is a polynomial" was not brought up spontaneously, then the interviewer asked it.

Two. How does one check whether $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ and $b_n x^n + b_{n-1} x^{n-1} + ... + b_1 x + b_0$ are equal? (In some cases, when the student could not relate to the general expression of polynomials, the question was repeated with explicit examples, such as $4x^5 + 3x + 1$ and $4x^5 - 2x + 1$).

Three. Is $x^4 + x^3 + 7$ a vector?

Four. What is a vector?

Responses to this question were organized according to the following aspects:

- What is a polynomial?
- Equality of polynomials.
- What is a vector?
- The confrontation of contradicting interpretations of vector and polynomial.

I will start with a lengthy analysis of a single student's interview.

Harve - What is a polynomial?

Int.: *What is this*? [Points at the written polynomial].

Har.: *A polynomial.*

Int.: *What is a polynomial*?

Har.: *A polynomial is a… an equation. Wait, actually it is not an equation, a polynomial is un… addition of…, with un,…and, well,… I want to say, no, at the beginning I wanted to say equation, but I don't have any equality-sign here, so I dropped it.*

Int.: *Well.*

Har.: *Eh, it's an expression, with, eh, of the deg, of the deg, of some particular degree. From degree two up it can be a polynomial.*

Harve – Equality of polynomials – A.

Int.: *O.K., just give me a minute* [writes: $4x^5 + 3x + 1$ $4x^5 - 2x + 1$]. *Here are two polynomials.*

Harv.: *Ehem.*

Int.: *Are they equal*?

Harv.: [Thinks.]

Int.: *Equal or unequal*?

Harv.: *Maybe they are equal, when x equals zero. Only then they will be equal. Yes, then I…, Actually no, I don't understand…Ah, the x, yes, the x must be zero, for them to be equal. Any other digit…*

Harve – Equality of polynomials – B.

Int.: *But the polynomial, the polynomial itself.*

Harv.: *Polynomial to polynomial*?

Int.: *Yes*.

Harv.: *What does it mean, polynomial to polynomial? And what, you need…*

Int.: *What do you need to do in order to check, whether they are equal. That is the question.*

Harv.: *This means to start what, to reduce between them, to make equal* [writes an equality sign = between the two polynomials: $4x^5 + 3x + 1 \quad = \quad 4x^5 - 2x + 1$].

…

To find the x or something like that? No? …To find what is the x itself, which maybe they will be equal.

Harve's relation to the polynomials brings into mind the *Action* conception of function, described within the APOS theoretic perspective:

> *An* ***action*** *is a repeatable mental or physical manipulation of objects. Such a conception of function would involve, for example, the ability to plug numbers into an algebraic expression and calculate. (Dubinsky and Harel, 1992)*

Within this perspective we also have a detailed description of the limitations of the action conception of function. Applied to polynomials, we can anticipate that a student whose conception of polynomial is limited to action conception of function, would probably be able to calculate (component per component) a linear combination of two polynomials, but will not be able to discuss and investigate characteristics of operations such as polynomial addition or multiplication by scalar. Hence he or she will find it difficult to consider whether a given set of polynomials is or is not a vector space.

Now we can sum up what we know about Harve's conception of a polynomial:

- A polynomial is not an *Equation*;
- It is an *Expression*;
- His function concept is at the *Action* level of development – far from the required level of *object*. This is especially evident when he asks: *What does it mean, polynomial to polynomial*? As if for him, polynomials are not comparable objects.
- Equality: Point-wise Equality, and for but some substitutions.

Let us look at Harve's responses in relation to the other aspects.

<u>Harve – Is a polynomial a vector?</u>

Int.: *Is the expression that we had here*, [reads and points at $\mathbf{x^4 + x^3 + 7}$], *is it a vector*?

Harv.: [Thinks]…*A vector needs to have a size and a direction*. [Thinks] *And here…* [sighs], *I can't even turn it into a vector, what should I do, x to the power of 4 and x to the power of 3* [writes: $\begin{pmatrix} \mathbf{x^4} \\ \mathbf{x^3} \\ \mathbf{7} \end{pmatrix}$] *and sev* [illegible] *not, it does not seem right*.

<u>Harve – What is a vector?</u>

Int.: *O.K. So for you a vector is,… So what is a vector*?

Harv.: *A vector is a number that has both a size, its size, and the direction. That means, to which direction does it move…*

So for Harve, a vector is either a mathematical *thing* that has size and direction, or else, maybe a tuple.

Having analyzed meticulously the responses of one particular student, I will present other students' responses accumulated according to the suggested categories.

What is a polynomial.

The students related to three different interpretations of polynomials:

- Polynomials as equations
- Polynomials as meaningless expressions.
- Polynomials as functions.

Polynomials as equations.

Some students thought that polynomials were equations. Here are some such answers to the question *What is this* [$\mathbf{x^4+x^3+7}$]?

Kid: *An equation.*

Guil: *Does it not have to be equal to zero?*

Jul: *This is an equation.*

Others, after considering this interpretation, rejected it. They concluded that $\mathbf{x^4+x^3+7}$ was *not* and equation:

Harve: *A polynomial is a… an equation. Wait, actually it is not an equation*

And then:

…at the beginning I wanted to say equation, but I don't have any equality-sign here, so I dropped it.

Ala: *It is not an equation, as if, it is not equal to anything*

Mad: *It is not an equation*.

Polynomials as meaningless expressions:

Hersch: *A polynomial is…, a set of elements*

Ala: *…It is some exercise*.

And then:

Are these powers? 3 and 4 [Points at them]?

Int.: *Yes, it is* [reads] $\mathbf{x^4+x^3+7}$.

Ala.: *Then it is an exercise*

Michel: *x to the power of 4 and …a number, two variables and a number*.

Exercise was considered in this category as I think that by this term students referred to some combination of mathematical symbols to be manipulated according to some syntactic rules.

Polynomials as functions

1st. Function as an input-output mechanism (*action* level in the development of the concept):

Harv. About $4x^5+3x+1$ and $4x^5-2x+1$:

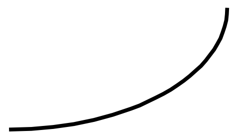
Ah, the x, yes, the x must be zero, for them to be equal. Any other digit…

Mad said About $\mathbf{x^4+x^3+7}$:

If you substitute any number, it gives a result. … Let's say 7, gives us 7. Zero, seven. … …one is nine.

2nd. Function as a graph

Joel: *When I have, when I have a polynomial* [says and writes:] $\mathbf{x^2 + x +}$ *it looks like this* [draws:]



Joel was the only student to present a graphic interpretation of function.

<u>The coefficient criterion for equality of polynomials</u>
<u>A. Knowledge</u>

Lin is one example of a student who can be considered to actually know this criterion:

Int.: *Here are written letters, but suppose you had numbers, how would you check?*
Lin: *I should have, I would, would compare, ah, greatest power to greatest power*.
Int.: *Even if the greatest power here was 5 and the greatest power here was 3, you would…*
Lin: *No, no, no, the meaning is if the power here 5, as I said, and here 5, then I…*
Int.: *So what would you compare*?
Lin: *The…, if the powers were equal then I'd compare the numbers*.
Int.: *They are called coefficients*.
Lin: *Coefficients. And they are equal then they are equal, If they are not equal…*
Int.: *Wait, wait, if they are equal with the high powers then you can stop checking*?
Lin.: *No. no. I am speaking about, for example here it was 5 and 5* [points at both given polynomials, in letters]. *I'd look, 5, 5. And I'd look and see that the numbers are equal, that the coefficients are identical, then I'd move to the lower power, to 4 or 3, depends what, what was there.*

Why do I categorize this response under knowledge? Here I am using a description of knowledge used often by researchers who work within the theoretic perspective APOS:

> *A person's mathematical knowledge is her or his tendency to respond to certain kinds of perceived problem situations by constructing, reconstructing and organizing mental processes and objects to use with the situation. (*Dubinsky, E., 1989).

We might say that Lin did reconstruct and describe an action-scheme which uses the coefficient criterion for the check of the equality of two polynomials.

5 (out of 12) students explained a proper version of the coefficient criterion, and could be categorized as knowers.

<u>The coefficient criterion for equality of polynomials</u>
<u>B. An enlarged criterion</u>

Mad was another student who expressed knowledge (proper use) of the Coefficient Criterion, but he went further to enlarge this criterion into an invention of a kind of "order relation" between polynomials:

Int.: *Suppose these are two polynomials and we wrote here letters instead of the coefficients. Yes? If theses were numbers, how would you check if these two polynomials were equal*?

Mad: *You subtract this from that, if it equals zero, they are equal*.

Int.: *How subtract*?

Mad: *Here is degree 4, and here 4* [points at the appropriate components] *you take the coefficient of this minus the coefficient of that, the coefficients, if it is negative then this is smaller than that. If it is zero than they are equal, if it is positive then this is larger*.

<u>What is a vector?</u>

In order to analyse the student's ideas about polynomials and vectors, I will first present their ideas of what a vector was. First some concept images (Vinner, 1983)

1st. <u>*Size and direction*</u>

Harve: *A vector needs to have a size and direction.*

Hersch: *O.K., we said that the definition of vector is* **not** *something that has direction and size, which is what it usually is, so if not, every number, every mathematical operation, any part is a vector,… I don't have a definition.*

That is an example of a student's awareness to the conflict between his concept image and concept definition (Vinner, 1983).

<u>B. *Comes out of zero*</u>

Mad: *Something that comes out of zero and goes up to some point*.

Tania*: It is an axis, that comes out of the origin.*

<u>C. *Joins two points*</u>

Joel: *A vector…, it joins two points. … And it has a direction*.

<u>D. A tuple?</u>

Harv. [about $\mathbf{x}^4 + \mathbf{x}^3 + 7$, after saying that *A vector needs to have a size and a direction.*] *And here…* [sighs], *I can't even turn it into a vector, what should I do, x to the power of 4 and x to the power of 3* [writes]:

$$\begin{bmatrix} x^4 \\ x^3 \\ 7 \end{bmatrix}$$

and seven. It's not, it does not seem right.

<u>E. A element of a vector-space</u>

<u>Lin</u>

Int.*: Is this polynomial* [points at $\mathbf{x}^4 + \mathbf{x}^3 + 7$]…*a vector*?

Lin.: *Is this polynomial a vector*? [Thinks.]

Int.: *How do you know if something is a vector*?

Lin.: *If, eh, a vector is actually a sub, it needs to be a vector-space. If it fulfills all the rules of a vector-space, and if it is a vector-space, which I do not…*

Int.: *Ah, do you hesitate because you do not remember whether it is a vector-space*?

Lin.: *No, no, I, yes, I do not remember if it is a vector-space*.

Int.: *O.K.*

Lin.: *If it does…*

Int.: *If it were a vector-space*?

Lin.: *Then yes, it is a vector*.

This is an example of the systemic thinking, discussed by Sierpinska (2000) and Fischbein et al. (1995). It means that a student is able to analyze a mathematical concept from the point of view of a system comprised of elements of its own kind, and their transformations. For Piaget, such ability indicates the organization of the concept *vector* into an operation (Piaget, 1975, 1976, Piaget and Inhelder, 1971). In terms of APOS it also means that the concept *polynomial* has developed, in the student's mind, into an *object.*

<u>Confrontations between concepts</u>

The last of response categories I present deals with confrontations between the student's concept of *polynomial* and his concept of *vector.* Two examples:

<u>Mad</u>

Int.: *This, the polynomial we have started with* [points at $\mathbf{x}^4+\mathbf{x}^3+7$*]. Is it a vector*?

Mad: *If you substitute any number, it gives a result*.

Int.: *And that's why it is a vector*?

Mad: *Let's say 7, gives us 7. Zero, seven*.

Int.: *Ehem*.

Mad: *One is nine*.

Int.: *And that's why it is a vector*?

Mad: *Yes*.

Int.: *What is a vector*?

Mad: *It's, let's say, something that comes out of zero and goes up to some point*.

Int.: *And this* [points at the polynomial] *comes out of zero and goes up to some point*?

Mad: *Not in every case, only in the substitution of zero*.

Int.: *Ehem*.

Mad: *If you substitute zero gives us zero seven*.

Int.: *Ehem*.

Mad: *And this is not a vector*.

So we can see that for Mad, a vector is something that *comes out of 0,* while for polynomial he has an action concept of function (substitution). In the confrontation between the two he first thinks that *Yes, it is a vector*, because of (0,7), but finally he concludes that *this is not a vector*, perhaps because *Not in every case, only in the substitution of zero.*

<u>Joel</u> contributes our second example of confrontation. His is a confrontation between two graphical concept images, that of a polynomial as a graph of the function, and that of a vector as an arrow that joins two points. We have quoted him before in relation to each of these concepts separately. Here is his full discussion of $\mathbf{x}^4+\mathbf{x}^3+7$ both as vector and polynomial:

Int.: *Is this polynomial* [points at $\mathbf{x}^4 + \mathbf{x}^3 + 7$] a vector?

Joel.: *Is it a vector? Ah, yes, it is a vector, yes*.

…

Int.: *What is called "vector"*?

…

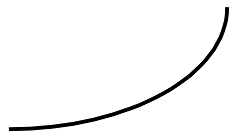
Joel.: *A vector…, it joins two points*.

Int.: *A polynomial joins two points?* [Points at the polynomial.]

Joel.: *Ah*?

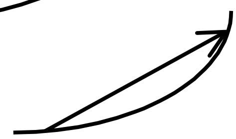
Int.: *Does a polynomial join two points*?

Joel.: *When I have, when I have a polynomial* [says and writes:] $\mathbf{x}^2 + \mathbf{x} +$ *it looks like this* [draws:]



Int.: *Ehem*

Joel.: *But this is not a vector* [adds a cord with an arrow:]



But the vector is between two points.

Int.: *So wait, wait, the arrow is a vector because it joins two points*.

Joel.: *And it has a direction*.

Int.: *O.K. So why is the polynomial a vector*?

Joel.: [Thinks] *Why is the polynomial a vector? Good question. I need to think of it*.

Conclusions

In a subject matter as difficult as linear algebra, even the "simplest" objects, polynomials, which are supposed to serve as "familiar" examples of the more abstract ideas, turn out to be interpreted in many different ways, both in the mathematics and in the students' minds. The consolidation of these interpretations poses problems for both teacher and student. How could research help us here?

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