$\boxed{7.5}$

③ $\int_S z\,dS$ όπου $S=\{(x,y,z): z=\sqrt{\alpha^2-x^2-y^2}$

$f(x,y,z)=z$, $T(x,y)=(x, y, \sqrt{\alpha^2-x^2-y^2})$

$T_x=\left(1, 0, \frac{-x}{\sqrt{\alpha^2-x^2-y^2}}\right)$, $T_y=\left(0, 1, \frac{-y}{\sqrt{\alpha^2-x^2-y^2}}\right)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{\sqrt{\alpha^2-x^2-y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{\alpha^2-x^2-y^2}} \end{vmatrix} =$$

$$= \left(\frac{x}{\sqrt{\alpha^2-x^2-y^2}}, \frac{y}{\sqrt{\alpha^2-x^2-y^2}}, 1\right)$$

$$\|T_x \times T_y\| = \sqrt{\frac{x^2+y^2}{\alpha^2-x^2-y^2}+1} = \frac{\alpha}{\sqrt{\alpha^2-x^2-y^2}}$$

Άρα $\iint_D \cancel{\sqrt{\alpha^2-x^2-y^2}} \cdot \frac{\alpha}{\cancel{\sqrt{\alpha^2-x^2-y^2}}}\,dx\,dy =$

πολικές $= \int_0^{2\pi}\int_0^{\alpha} \alpha \cdot r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{\alpha r^2}{2}\right]_0^{\alpha} d\theta = \pi\alpha^3$

(D ο δίσκος στο xy επίπεδο: $x^2+y^2\leq\alpha^2$)

Scanned with CamScanner

(5) $\int_S xyz\, dS$ όπου S το τρίγωνο με κορυφές

$(1,0,0)$, $(0,2,0)$, $(0,1,1)$

Εξίσωση επιπέδου $\quad \alpha x + by + cz + d = 0$

$$\left.\begin{array}{ll} (1,0,0): & \alpha + d = 0 \\ (0,2,0): & 2b + d = 0 \\ (0,1,1): & b + c + d = 0 \end{array}\right\} \underline{2x+y+z=2}$$

άρα έχω την παραμέτρηση:

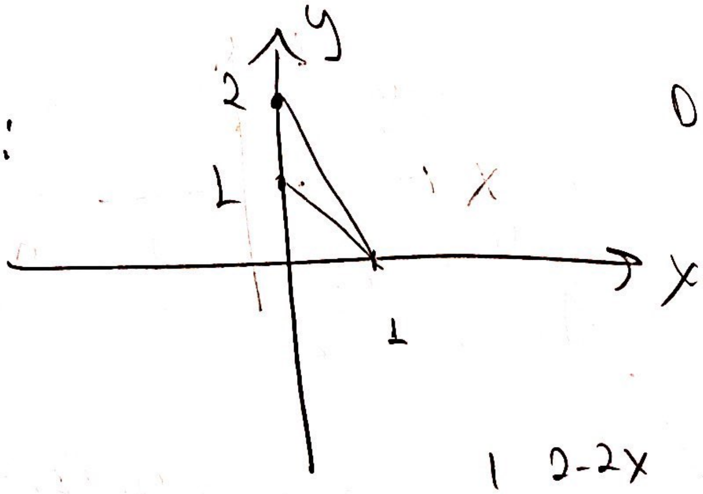
$$T(x,y) = (x, y, 2-2x-y)$$

$T_x = (1,0,-2)$, $T_y = (0,1,-1)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = (2,1,1)$$

$$\|T_x \times T_y\| = \sqrt{4+1+1} = \sqrt{6}$$

D:



$D: \quad 0 \le x \le 1$
$\quad\; 1-x \le y \le 2-2x$

άρα $$\int_S xyz\, dS = \int_0^1 \int_{1-x}^{2-2x} \sqrt{6}\, xy(2-2x-y)\, dy\, dx =$$

$$= \sqrt{6} \int_0^1 \int_{1-x}^{2-2x} 2xy - 2x^2y - xy^2 \, dy\, dx =$$

$$= \sqrt{6} \int_0^1 \left[y^2x - y^2x^2 - \frac{xy^3}{3} \right]_{1-x}^{2-2x} dx =$$

$$= \sqrt{6} \int_0^1 (2-2x)^2 x - (2-2x)^2 x^2 - \frac{x(2-2x)^3}{3} - (1-x)^2 x + (1-x)^2 x^2 + \frac{(1-x)^3 x}{3} dx$$

$$= \sqrt{6} \cdot \frac{1}{30} = \frac{\sqrt{6}}{30}.$$

(7) Υπολογίστε το $\int_S z\, dS$ όπου S η επιφάνεια

$z = x^2 + y^2, \quad x^2 + y^2 \leq 1$

$T(x,y) = (x, y, x^2+y^2)$

$T_x = (1, 0, 2x), \quad T_y = (0, 1, 2y)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1)$$

$\| T_x \times T_y \| = \sqrt{4x^2 + 4y^2 + 1}$

$$\int_S z\, dS = \iint_D (x^2+y^2) \sqrt{4x^2+4y^2+1} \, dx\, dy$$

όπου D: ο δίσκος $x^2 + y^2 \leq 1$

Πολικές: $\int_0^{2\pi} \int_0^1 r^3 \sqrt{4r^2+1} \, dr\, d\theta =$

$$= \int_0^{2\pi} \left(\left[\frac{2(4r^2+1)^{3/2}}{8 \cdot 3} r^2 \right]_0^1 - \int_0^1 \frac{(4r^2+1)^{3/2} \cdot 2r\, dr}{12} \right) d\theta$$

Scanned with CamScanner

$$= \int_0^{2\pi} \frac{5^{3/2}}{12}\, d\theta - \int_0^{2\pi} \left[\frac{2}{5} \frac{(4r^2+1)^{5/2}}{4 \cdot 12} \right]_0^1 d\theta =$$

$$= \frac{5^{3/2}\pi}{6} - \int_0^{2\pi} \frac{5^{5/2}}{20} - \frac{1}{120} =$$

$$= \frac{5\sqrt{5}}{6}\pi - \frac{(\sqrt{5})^5}{10}\pi + \frac{\pi}{60}$$

$$= \frac{50\sqrt{5}\pi - 25\sqrt{5}\pi}{60} + \frac{\pi}{60} = \frac{25\pi\sqrt{5}}{60} + \frac{\pi}{60} =$$

$$= \pi \frac{(25\sqrt{5}+1)}{60}$$

Scanned with CamScanner

(2) $\varphi: D \subset \mathbb{R}^2 \to \mathbb{R}^3$

$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

α) $\frac{\partial \varphi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \to T_u$, $\frac{\partial \varphi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \to T_v$

$E = \left\|\frac{\partial \varphi}{\partial u}\right\|^2$, $F = \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \varphi}{\partial v}$, $G = \left\|\frac{\partial \varphi}{\partial v}\right\|^2$

νδο $A(S) = \int_D \sqrt{EG - F^2}\, du\, dv$

α) $A(S) = \int_D \|T_u \times T_v\|\, du\, dv$

$*$ $\|T_u \times T_v\|^2 = (T_u \times T_v) \cdot (T_u \times T_v) =$

$= (T_u \cdot T_u)(T_v \cdot T_v) - (T_u \cdot T_v) \cdot (T_v \cdot T_u) =$

$= \left\|\frac{\partial \varphi}{\partial u}\right\|^2 \left\|\frac{\partial \varphi}{\partial v}\right\|^2 - \frac{\partial \varphi}{\partial u} \frac{\partial \varphi}{\partial v} \cdot \frac{\partial \varphi}{\partial u} \frac{\partial \varphi}{\partial v} =$

$= EG - F^2$

άρα $\|T_u \times T_v\| = \sqrt{EG - F^2}$

άρα $A(S) = \int_D \sqrt{EG - F^2}\, du\, dv$

$\int_S f\, dS = \int_D f(\varphi(u,v)) \sqrt{EG - F^2}\, du\, dv$

Scanned with CamScanner

b) Αν $\frac{\partial \varphi}{\partial u}$, $\frac{\partial \varphi}{\partial v}$ ορθογώνια

$$\frac{\partial \varphi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} = 0 \Rightarrow F = 0$$

άρα $A(\varphi) = \int_D \sqrt{EG} \, du\, dv$

Scanned with CamScanner

④ $\int_S (x+y+z)\,dS$, $S: (x,y,z)$ με $x^2+y^2+z^2=1$

$$\int_S (x+y+z)\,dS = \int_S x\,dS + \int_S y\,dS + \int_S z\,dS =$$

$$= 3\int_S z\,dS$$

$$T(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$$

$$\|T_\theta \times T_\varphi\| = |\sin\varphi|$$

άρα $3\int_S z\,dS = 3\int_0^{2\pi}\int_0^{\pi} \sin\varphi\cos\varphi\,d\varphi\,d\theta =$

$$= 3\int_0^{2\pi}\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi} d\theta = 3\int_0^{2\pi} 0 = 0.$$

Scanned with CamScanner

(15) $J(\varphi) = \frac{1}{2}\int_D \left(\left\| \frac{\partial \varphi}{\partial u} \right\|^2 + \left\| \frac{\partial \varphi}{\partial v} \right\|^2 \right) du\, dv$

νδο $A(\varphi) \leq J(\varphi)$ και η ισοτητα ισχυει

ανν α) $\left\| \frac{\partial \varphi}{\partial u} \right\|^2 = \left\| \frac{\partial \varphi}{\partial v} \right\|^2$ και b) $\frac{\partial \varphi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} = 0$

Εχουμε οτι $\left(\left\| \frac{\partial \varphi}{\partial u} \right\| - \left\| \frac{\partial \varphi}{\partial v} \right\| \right)^2 \geq 0$

$$\Rightarrow \left\| \frac{\partial \varphi}{\partial u} \right\|^2 - 2 \left\| \frac{\partial \varphi}{\partial u} \right\| \left\| \frac{\partial \varphi}{\partial v} \right\| + \left\| \frac{\partial \varphi}{\partial v} \right\|^2 \geq 0$$

$$\Rightarrow \left\| \frac{\partial \varphi}{\partial u} \right\|^2 + \left\| \frac{\partial \varphi}{\partial v} \right\|^2 \geq 2 \left\| \frac{\partial \varphi}{\partial u} \right\| \left\| \frac{\partial \varphi}{\partial v} \right\| \quad (1)$$

$$A(\varphi) = \int_D \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| du\, dv$$

$$\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| = \left\| \frac{\partial \varphi}{\partial u} \right\| \left\| \frac{\partial \varphi}{\partial v} \right\| |\sin\theta| \overset{(1)}{\leq} \frac{1}{2} \left(\left\| \frac{\partial \varphi}{\partial u} \right\|^2 + \left\| \frac{\partial \varphi}{\partial v} \right\|^2 \right.$$

Scanned with CamScanner

άρα $A(\varphi) \leq J(\varphi)$

Για να έχω ισότητα, δηλ.

$$\left\|\frac{\partial \varphi}{\partial u}\right\| \left\|\frac{\partial \varphi}{\partial v}\right\| |\sin\theta| = \frac{1}{2}\left(\left\|\frac{\partial \varphi}{\partial u}\right\|^2 + \left\|\frac{\partial \varphi}{\partial v}\right\|^2\right)$$

πρέπει $|\sin\theta| = 1 \Rightarrow \cos\theta = 0$ άρα $\frac{\partial \varphi}{\partial u} \perp \frac{\partial \varphi}{\partial v}$

$$\Rightarrow \frac{\partial \varphi}{\partial u} \cdot \frac{\partial \varphi}{\partial v} = 0$$

και $\left\|\frac{\partial \varphi}{\partial u}\right\| = \left\|\frac{\partial \varphi}{\partial v}\right\|$

Scanned with CamScanner