THE GEOMETRY OF COMPLEX HYPERBOLIC PACKS

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ABSTRACT. Complex hyperbolic packs are 3-hypersurfaces of complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ which may be considered as dual to the well known bisectors. In this article we study the geometric aspects associated to packs.

1. Introduction and Statement of Results

This article is devoted to the study of a rather new family of 3–hypersurfaces of complex hyperbolic plane, called *packs*. Our study is divided in two overlapping parts. In the one part we discuss in extent the differential geometry of packs, by simultaneously highliting its similarities as well as its differences to the geometry of the well known *bisectors*. Secondly, and from the study of the action of the symplectic group of complex hyperbolic plane on packs, we construct in detail a quasiconformal mapping of the Heisenberg group associated to packs. This mapping is suggestive towards the further study of quasiconformal deformations of fundamental domains in complex hyperbolic plane.

Complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ is a 2–complex dimensional complete Kähler manifold with constant holomorphic sectional curvature -1 and real sectional curvature pinched between $-1/4$ and -1. Its group of holomorphic isometries is $\mathrm{PU}(2,1)$. Real hyperbolic plane is embedded into $\mathbf{H}^2_{\mathbb{C}}$ in two ways. First as a complex submanifold (a copy of the Poincaré disk model $\mathbf{H}^1_{\mathbb{C}}$) and secondly as a totally real Lagrangian submanifold (a copy of the Beltrami–Klein model $\mathbf{H}^2_{\mathbb{R}}$). Complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ has no totally geodesic hypersurfaces of codimension 1, unlikely to the case of its real counterpart, the hyperbolic space $\mathbf{H}^4_{\mathbb{R}}$. Nevertheless, a well known class of fair substitutes exist; these are the so called bisectors. The study of bisectors goes back to the 1930's in the work of Giraud and an extensive treatment on the subject may be found in [6]. Below we state in brief some well known facts about bisectors, see [6] and also Section 2.3.2 for more details. Following Mostow [14], a bisector B may be obtained as follows. Let γ be a geodesic in $\mathbf{H}^2_{\mathbb{C}}$ and denote by a and r its endpoints in $\partial\mathbf{H}^2_{\mathbb{C}}$. Let L be the unique complex geodesic spanned by a and r and consider the projection $\Pi_L : \mathbf{H}^2_{\mathbb{C}} \to L$. Then

$$B = B(\gamma) = \bigcup_{x\in\gamma} \Pi_L^{-1}(x).$$

Hence B is a 3–hypersurface of $\mathbf{H}^2_{\mathbb{C}}$ foliated by complex lines $\Pi_L^{-1}(x)$, $x \in \gamma$. This is called the *slice* decomposition of a bisector and from that we obtain an integrable CR structure of codimension 1 for each bisector. Goldman showed (see Theorem 5.1.10 in [6]) that bisectors also admit another decomposition (the *meridianal* decomposition), this time into Lagrangian planes which all intersect at γ. Since a bisector $B = B(\gamma)$ only depends on the edpoints of γ and $\mathrm{PU}(2,1)$ acts (doubly) transitively in $\mathbf{H}^2_{\mathbb{C}}$, we immediately have that $\mathrm{PU}(2,1)$ acts transitively in the space of bisectors. Considering its actual differential geometric properties, a bisector B is a minimal hypersurface

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of complex hyperbolic plane with zero Gauss–Kronecker curvature (see for instance [6] or, for a somewhat different treatment see [7]).

The natural counterparts of bisectors are packs; these were introduced by Will in [17] and in their general form by Parker and Platis in [15]. Roughly speaking, packs are 3–hypersurfaces of complex hyperbolic plane which are naturally foliated by Lagrangian planes. That is, a pack may be visualised as an infinite deck of cards, each of which is a Lagrangian plane. A pack P may be obtained by a loxodromic element C of the isometry group, see Section 3 below for details. But unlikely to the case of bisectors, the isometry group of $\mathbf{H}^2_{\mathbb{C}}$ does not necessarily act transitively in packs; transitive action depends on a parameter associated to each pack, called the curl factor κ of the pack which is a measure of the rotation of each Lagrangian plane around the axis of C. This factor constitutes the main obstruction for the transitive action according to the following Theorem.

Theorem 3.5. *Two packs P_1 and P_2 with curl factors κ_1 and κ_1 respectively are isometric if and only if $\kappa_1 = \kappa_2$.*

This is a rather restrictive Theorem in terms of the action of the isometry group. In order to obtain a non restrictive result, we can only turn to the action of the group of symplectic diffeomorphisms. But before that, we can make a statement about the CR geometry of packs and their "slice" decomposition. By our Proposition 3.12, a pack $P = P(C)$ with curl factor κ admits a singular codimension 1 foliation which is such that its singular leaf is the complex axis of C and each non singular leaf is biholomorphic to the Riemann surface

$$w = z^{\frac{1-3\kappa i}{2}}.$$

We now turn our discussion to the action of the group of symplectic diffeomorphisms of complex hyperbolic plane in the space of packs; this is our first main result:

Theorem 5.1. *The group of symplectomorphisms $\mathbf{Sp}(\mathbf{H}^2_{\mathbb{C}})$ of complex hyperbolic plane acts transitively in packs.*

We are able to give a quite simple proof of this Theorem by making use of cylindrical coordinates for complex hyperbolic plane which we use throughout the whole paper, see Section 4. The idea of the proof of Theorem 5.1 is to construct a symplectomorphism of $\mathbf{H}^2_{\mathbb{C}}$ which maps a certain pack with zero curl factor (a *flat* pack) to another certain pack with curl factor κ, see Lemma 5.3. From the proof of this Lemma, another geometrical object arises which is fair to call a $\kappa-$bisector. In fact we show that to each bisector B and to each real number κ there are associated exactly two such manifolds which are equidistant to B in distance depending only on κ. This generalises the hyperbolic geometric notion of equidistant lines, to the complex hyperbolic setting. One should also think of $\kappa-$bisectors as submanifolds looking very much alike to bisectors but which, instead of having a geodesic as their real spine, they rather have a horocycle, see for details Section 5.1.

To study the Riemannian aspects of packs, we follow the line suggested by Goldman in [6] for the case of bisectors. But here we have modified his method motivated by Theorem 7.2, see Section 7. In this way, we are able to give comparative results concerning packs and bisectors.

Theorem 7.5. *Let P be a pack with curl factor κ and B be a $\kappa-$bisector. Then*

(1) *P is a minimal submanifold of $\mathbf{H}^2_{\mathbb{C}}$ with zero Gauss-Kronecker curvature and*
(2) *B is a submanifold of $\mathbf{H}^2_{\mathbb{C}}$ with zero Gauss-Kronecker curvature. Moreover, it is minimal if and only if $\kappa = 0$.*

Before we state our last main theorem, which relates packs to quasiconformal mappings of the Heisenberg group, we wish to recall in brief some facts concerning bisectors and *complex hyperbolic quasi–Fuchsian space*. In this way, our motivation will be entirely transparent. Bisectors have been used to construct fundamental polyhedra for the action of a subgroup of the isometry group in $\mathbf{H}^2_{\mathbb{C}}$; in particular, Dirichlet domains have boundaries comprising of pieces of bisectors. Thus, bisectors are directly related to the study of the space of complex hyperbolic quasi–Fuchsian representations $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ of the fundamental group π_1 of a topological surface Σ of negative Euler characteristic. We recall that $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ is the set of discrete, faithful, type preserving and geometrically finite representations of π_1 into the isometry group of complex hyperbolic plane, that is, it is the natural generalisation of the well known Teichmüller space $\mathcal{T}(\Sigma)$ of Σ to the complex hyperbolic setting. For more details, see [15] and the references stated therein.

Teichmüller space $\mathcal{T}(\Sigma)$ comprises of Fuchsian representations of π_1, that is discrete, faithful, type preserving and geometrically finite representations of π_1 into $\mathrm{SU}(1,1)$. Every such representation is then homotopically equivalent to a quasiconformal representation; in this case, by uniformisation we may identify π_1 with a Fuchsian group Γ and then any $\rho \in \mathcal{T}(\Sigma)$ is equivalent to one of the form

$$\rho(\gamma) = f_\mu \circ \gamma \circ f_\mu^{-1}$$

where f_μ is a quasiconformal self mapping of the hyperbolic plane which solves the Beltrami equation $\partial f/\partial \overline{z} = \mu \partial f/\partial z$ for some $\Gamma-$invariant Beltrami coefficient μ and fixes three points in the boundary $\mathbb{R} \cup \{\pm\infty\}$.

One of the most distinguished problems in the study of complex hyperbolic quasi–Fuchsian space $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ is to determine if the analogous result holds there, i.e. whenever a representation ρ in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ is homotopically equivalent to a quasiconformal representation. This is not at all trivial problem; should we wish to solve this problem, then in principle we have to construct a fundamental domain for an arbitrary $\Gamma = \rho(\pi_1)$ and then deform it inside $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ in a way such that all induced representations are quasiconformal. In other words, we have to construct a curve $\rho_t \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$, with $\rho_0 = \rho$ and t small enough, and then show that for each t there exists a homeomorphism f_t of the closure of complex hyperbolic plane which is quasiconformal on the boundary and which is ρ_t-equivariant:

$$\rho_t(\gamma) \circ f_t = f_t \circ \gamma, \ \gamma \in \Gamma_0 = \rho(\pi_1).$$

A primary obstacle is that not all $\rho \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$ look alike; their nature is determined by a natural invariant associated to each representation, called the *Toledo invariant*. We shall not discuss the details about this invariant here. The interested reader should consult the introduction of [15] for further details.

However, and by a general Theorem of Guichard, [8], it follows that around each $\rho \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$ there exists an open neighborhood of ρ comprising only of elements of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. (See also [15] for an entirely different proof of the result in a special case).

The tool for the solution of the general problem is the theory of quasiconformal mappings of the Heisenberg group $\mathcal{H}$ as this was developed by Korányi and Reimann (see for instance [12] and [13]). This theory is the analogous to our case, of the Ahlfors–Bers theory of quasiconformal mappings of the complex plane and we have to use it in order to associate a quasiconformal deformation of the Heisenberg group to a deformation of a fundamental polyhedron. To that direction little progress has been made so far; for instance, Aebischer and Miner proved this result for the elementary case of complex hyperbolic quasi–Fuchsian space of a classical complex hyperbolic Schottky group of n generators, see [1]. Such a group admits a fundamental domain whose sides are disjoint bisectors. (For the definition and properties of the classical Schottky groups, see [16]). Their line of proof is based on the construction of a quasiconformal deformation of the Heisenberg group associated to families of bisectors which form the boundaries of the deformed initial fundamental domain. Therefore one should ask if their construction can be extended in the general case. Unfortunately, this does not seem to be likely in all cases. For instance, bisectors behave rather badly at $\mathbb{R}-$Fuchsian points of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$, that is representations in $\mathrm{SO}(2,1) < \mathrm{SU}(2,1)$. Thus it is reasonable to turn our attention to packs. In fact, let $\{P_t\}$, $t \in [0,1]$ be a given family of packs. We only assume that this family is smooth enough in t. Then, from Theorem 5.1 we are able to construct a quasiconformal deformation of $\mathfrak{H}$ which is associated to the family $\{P_t\}$.

Theorem 6.4. *Let $\{P_t\}$, $t \in [0,1]$ be a $\mathcal{C}^1$ family of packs. Then the following hold.*

1. *Associated to $\{P_t\}$ there is a continuously time dependent Hamiltonian vector field $\mathbf{b}_t$ in $\mathbf{H}^2_{\mathbb{C}}$ generating a flow of symplectomorphisms $\phi_{s,t}$ such that $\phi_{0,t}(P_0) = P_t$.*
2. *The Hamiltonian function B_t of $\mathbf{b}_t$ is $\mathcal{C}^\infty$ and continuous in t.*
3. *$\mathbf{b}_t$ extends smoothly to $\overline{\mathbf{H}^2_{\mathbb{C}}}$ as a flow of contactomorphisms.*
4. *Let $M = \max_{t\in[0,1]} |\kappa'(t)| \geq 0$. Then the family $\phi_{s,t}$ is quasiconformal and its dilation satisfies the following inequality.*

$$\| \mu_t \|^2_\infty \leq \tanh\left(\frac{3\sqrt{2}}{2}\, M\right).$$

The reader should compare this Theorem to Lemma 2.1 in [1]; the latter is the necessary step for the proof of the main result of the paper (See Theorems 1.1 and 1.3 in [1]). Using our Theorem 6.4 we are able to prove exactly the same result for the complex hyperbolic quasi-Fuchsian group of a complex hyperbolic $p-$Scottky group of n generators. Such a group admits a fundamental domain whose boundary consists of packs rather than bisectors. A proof of this result will appear elsewhere.

This paper is organised as follows. In Section 2 we present the background material we use for our further discussion. In Section 3 we deal with packs and some of their intrinsic geometric properties. Theorem 3.5 is proved in Section 3.1. A representation for the standard pack with curl factor κ is given in Section 3.1.1 and this is used to study the CR geometry of packs in Section 3.2. In Section 4 we introduce the cylindrical model for complex hyperbolic plane. In Section 5 we prove that the group of symplectomorphisms of $\mathbf{H}^2_{\mathbb{C}}$ acts transitively in packs (Theorem 5.1). Section 6 is devoted to the proof of Theorem 6.4. This is carried out after extending the cylindrical coordinates to the boundary of $\mathbf{H}^2_{\mathbb{C}}$ (Section 6.1). Finally, in Section 7 we present a simultaneous study of the differential geometry of bisectors and packs proving Theorems 7.2 and 7.5.

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2. Preliminaries and Background Material

Most of the results stated in this section are well known. The reader may consult the reference book of Goldman [6], for more information about the standard aspects of complex hyperbolic plane presented in Sections 2.1, 2.2 and 2.3. In particular, for Section 2.2.1 the reader may consult [15]. Section 2.4 is not really needed for our construction, but we have chosen to add it in order to further clarify the context. The interested reader may consult for instance Chapter 7 of [4] for an extensive presentation of quasiconformal symplectomorphisms. For Section 2.5 we refer the reader to the standard bibliography about the Heisenberg group and its contact structure. For more information about sections 2.5.1 and 2.5.2, one may consult Chapter 7 of [4] as well as [10], [11], [12] and [13].

2.1. **Complex hyperbolic plane.** Let $\mathbb{C}^{2,1}$ be the complex vector space $\mathbb{C}^3$ equipped with the non-degenerate, indefinite Hermitian form $\langle\cdot,\cdot\rangle$ of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w}\rangle = z_1\overline{w_3} + z_2\overline{w}_2 + z_3\overline{w_1}.$$

For all $\mathbf{z} \in \mathbb{C}^{2,1}$ we have that $\langle \mathbf{z}, \mathbf{z} \rangle \in \mathbb{R}$. We thus may define subsets V_-, V_0 and V_+ of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_- &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}, \\ V_0 &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0 \right\}, \\ V_+ &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0 \right\}. \end{aligned}$$

We say that $\mathbf{z} \in \mathbb{C}^{2,1}$ is *negative*, *null* or *positive* if $\mathbf{z}$ is in V_-, V_0 or V_+ respectively. The *Siegel domain model for complex hyperbolic plane* $\mathbf{H}^2_{\mathbb{C}}$ is defined as follows. Let $\mathbb{P} : \mathbb{C}^{2,1} \to \mathbb{CP}^2$ be the usual projection and let also the section defined by $z_3 = 1$. We consider what it means for $\langle \mathbf{z}, \mathbf{z} \rangle$ to be negative and thence we obtain $\mathbf{z} \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle \mathbf{z}, \mathbf{z} \rangle = 2\Re(z_1) + |z_2|^2 < 0.$$

Definition 2.1. The domain $\{z = (z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0\}$ is called the Siegel domain of $\mathbb{C}^2$ and forms the *Siegel domain model* of complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$. Its boundary is the paraboloid defined by $2\Re(z_1) + |z_2|^2 = 0$.

2.1.1. *Kähler structure.* Complex hyperbolic plane is a Kähler manifold with constant holomorphic sectional curvature -1 and real sectional curvature pinched between -1 and $-1/4$. Let

$$\rho(z) = -2\Re(z_1) - |z_2|^2 > 0 \tag{2.1}$$

be the defining function of $\mathbf{H}^2_{\mathbb{C}}$. The Bergman-Kähler symplectic form for $\mathbf{H}^2_{\mathbb{C}}$ is given by

$$\begin{aligned} \Omega &= 2i\partial\overline{\partial}(\log \rho(z)) \\ &= \frac{-2i}{\rho^2(z)} \left(dz_1 \wedge d\overline{z}_1 + 2i\Im(z_2 dz_1 \wedge d\overline{z}_2) - 2\Re(z_1) dz_2 \wedge d\overline{z}_2 \right). \end{aligned} \tag{2.2}$$

Accordingly, the Bergman metric tensor is given by

$$ds^2 = \frac{4}{\rho^2(z)} \left(|dz_1|^2 + 2\Re(z_2 dz_1 d\overline{z_2}) - 2\Re(z_1)|dz_2|^2 \right). \tag{2.3}$$

2.2. **Isometries.** The full group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, but we prefer to consider instead the group $\mathrm{SU}(2,1)$, that is the set of matrices which are unitary with respect to $\langle \cdot, \cdot \rangle$ and have determinant 1. The group $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$.

There exist three kinds of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$:

(i) *Loxodromic* isometries, each of which fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$. One of these points is attracting and the other is repelling.
(ii) *Parabolic* isometries, each of which fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$.
(iii) *Elliptic* isometries, each of which fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

In our context we deal only with loxodromic isometries.

2.2.1. *Loxodromic isometries.* For any $\lambda \in \mathbb{C}^* = \{\lambda \in \mathbb{C} : -\pi < \Im(\lambda) \leq \pi$ we define $E(\lambda) \in \mathrm{SU}(2,1)$ by

$$E(\lambda) = \begin{bmatrix} e^{\lambda} & 0 & 0 \\ 0 & e^{\overline{\lambda}-\lambda} & 0 \\ 0 & 0 & e^{-\overline{\lambda}} \end{bmatrix}. \tag{2.4}$$

Let

$$S = \left\{ \lambda \in \mathbb{C} : \Re(\lambda) > 0,\ \Im(\lambda) \in (-\pi, \pi] \right\}. \tag{2.5}$$

If $\lambda \in S$ then $E = E(\lambda)$ is a loxodromic map with attractive (resp. repelling) fixed point ∞ (resp. o). If $\Re(\lambda) < 0$ then $-\overline{\lambda} \in S$ and $E(\lambda)$ is a loxodromic map with attractive fixed point o and repelling fixed point ∞.

Let $C \in \mathrm{SU}(2,1)$ be a matrix representing a loxodromic isometry and also let $a, r \in \partial\mathbf{H}^2_{\mathbb{C}}$ be the attractive and the repelling fixed points of A with lifts $\mathbf{a}, \mathbf{r}$ to V_0 respectively. From the transitive action of $\mathrm{SU}(2,1)$ on $\partial\mathbf{H}^2_{\mathbb{C}}$ it follows that there exists a $Q \in \mathrm{SU}(2,1)$ whose columns are projectively $\mathbf{a}$, $\mathbf{n}$, $\mathbf{r}$, where $\mathbf{n}$ is a vector polar to $\mathbf{a}$ and $\mathbf{r}$. We may write

$$C = QE(\lambda)Q^{-1}, \tag{2.6}$$

where $E(\lambda)$ is given by (2.4). The geodesic (r, a) joining r and a is called the *real axis* of C and the complex line $L_{\mathbb{C}}$ spanned by a and r is called the *complex axis* of C. The trace of C is

$$\mathrm{tr}(C) = \tau(\lambda) = e^{\lambda} + e^{\overline{\lambda}-\lambda} + e^{-\overline{\lambda}}.$$

The complex number $\lambda(C) = l(C) + i\theta(C)$ is called the *complex hyperbolic length* of C. Its real part $l(C)$ is half the geodesic length of the real axis of C and $\theta(C)$ is half the rotation angle about the real axis.

2.3. **Submanifolds.**

2.3.1. *Totally geodesic submanifolds.* There exist only 2–dimensional totally geodesic submanifolds of $\mathbf{H}^2_{\mathbb{C}}$: a) Complex lines L which have constant curvature -1. These submanifolds realise isometric embeddings of $\mathbf{H}^1_{\mathbb{C}}$ into $\mathbf{H}^2_{\mathbb{C}}$. Every complex line L is the image under some $A \in \mathrm{SU}(2,1)$ of the complex line

$$L_0 = \{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : z_2 = 0\}. \tag{2.7}$$

b) Lagrangian planes R which have constant curvature $-1/4$. These in turn realise isometric embeddings of $\mathbf{H}^2_{\mathbb{R}}$ into $\mathbf{H}^2_{\mathbb{C}}$. Each Lagrangian plane R may be consider as the set of fixed points of an antiholomorphic inversion R of complex hyperbolic plane. The following holds, see [5].

Theorem 2.2. *An element $C \in \mathrm{SU}(2,1)$ is loxodromic if and only if it can be written as $C = I_2 \circ I_1$ where I_1 and I_2 are antiholomorphic inversions on disjoint Lagrangian planes R_1 and R_2 respectively.*

Every Lagrangian plane is the image under some element of $\mathrm{SU}(2,1)$ of the standard real Lagrangian plane $R_{\mathbb{R}}$; the latter admits the following normalisation.

$$R_{\mathbb{R}} = \left\{ \mathbf{r} = \begin{bmatrix} -e^{i\psi} \\ ir\sqrt{2\cos(\psi)}e^{i\psi/2} \\ 1 \end{bmatrix} \in V_- : (\psi, r) \in (-\pi/2, \pi/2) \times (-1, 1) \right\}. \tag{2.8}$$

This normalisation of $R_{\mathbb{R}}$ will be used in the proof of Proposition 3.10 below.

2.3.2. *Bisectors.* For an extensive study of bisectors, see [6]. Here we only state some basic facts about them. Let $z_1, z_2 \in \mathbf{H}^2_{\mathbb{C}}$ be two distinct points. The *bisector $B(z_1, z_2)$ equidistant from z_1 and z_2* is

$$B(z_1, z_2) = \{z \in \mathbf{H}^2_{\mathbb{C}} : \ \delta(z_1, z) = \delta(z_2, z)\}$$

where δ denotes complex hyperbolic distance. Let L be the complex line spanned by z_1 an z_2 and s be the real geodesic defined by

$$s(z_1, z_2) = B(z_1, z_2) \cap L.$$

Then L is called the *complex spine* and s is called the *real spine* of $B(z_1, z_2)$. The spine and the complex spine depend on the bisector and not to the points z_1, z_2. Let Π_L be orthogonal projection onto L. Then, following Mostow, the bisector B with spine s is the inverse image of s under Π_L. Thus, each slice of this bisector is the inverse image of a point of s under Π_L, and is a complex line. A bisector also admits a *meridianal decomposition*: it is the union of the Lagrangian planes which all meet in s, see Section 5.1.6 of [6]. Finaly, bisectors are completely determined by (the endpoints of) their spine s and we may thus write $B = B(s)$. Let

$$B_0 = \left\{(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}} : \Im(z_1) = 0\right\}. \tag{2.9}$$

If Π_0 is orthogonal projection onto L_0 where L_0 is as in 2.7, and s_0 is the geodesic $(-\infty, 0)$ then one may verify directly that

$$B_0 = \bigcup_{x\in s_0} \Pi_0^{-1}(x).$$

Thus B_0 is the bisector $B(s_0)$ which will be called the *standard bisector*. Suppose now that $B(s)$ is an arbitrary bisector with spine s and choose the element of the isometry group of $\mathbf{H}^2_{\mathbb{C}}$ mapping s onto s_0. Then from the discussion above we conclude that $B(s)$ is mapped onto B_0 and thus we obtain that the isometry group acts transitively on bisectors, see also section 5.2.1 of [6] for more details.

2.4. **Symplectomorphisms and complex dilation.** A *symplectomorphism* of $\mathbf{H}^2_{\mathbb{C}}$ is a diffeomorphism F such that $F^*\Omega = \Omega$. The group of symplectomorphisms $\mathbf{Sp}(\mathbf{H}^2_{\mathbb{C}})$ is an infinite dimensional Lie group. Denote by $\mathbb{J}$ is the natural complex structure of $\mathbf{H}^2_{\mathbb{C}}$. A symplectomorphism F defines another complex structure $\mathbb{J}_\mu$ in $\mathbf{H}^2_{\mathbb{C}}$ by the relation $\mathbb{J}_\mu = F_*^{-1} \circ \mathbb{J} \circ F_*$. By Lemma 7.5 of [4], there is a complex antilinear self mapping of the holomorphic tangent bundle $\mathrm{T}^{(1,0)}$ of the complex hyperbolic plane such that the holomorphic tangent bundle $\mathrm{T}^{(1,0)}_\mu$ of the $\mathbb{J}_\mu$ complex structure is

$$\mathrm{T}^{(1,0)}_\mu = \{Z - \overline{\mu Z} : Z \in \mathrm{T}^{(1,0)}\}.$$

The map μ is called the *complex dilation* of F. A neat description of the complex dilation is via a Beltrami system of equations, see pp. 401–402 of [4]. If dF be the Jacobian matrix of $F = (f_1, f_2)$ then there is a decomposition

$$dF = M_A + M_S = \begin{bmatrix} \Re(DF) & -\Im(DF) \\ \Im(DF) & \Re(DF) \end{bmatrix} + \begin{bmatrix} \Re(\overline{D}F) & \Im(\overline{D}F) \\ \Im(\overline{D}F) & -\Re(\overline{D}F) \end{bmatrix}$$

where

$$DF = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{bmatrix}, \quad \overline{D}F = \begin{bmatrix} \frac{\partial f_1}{\partial \overline{z_1}} & \frac{\partial f_1}{\partial \overline{z_2}} \\ \frac{\partial f_2}{\partial \overline{z_1}} & \frac{\partial f_2}{\partial \overline{z_2}} \end{bmatrix}.$$

The matrix of the complex dilation μ (denoted again by μ) is the 2×2 complex matrix $A + iB$ which is given by the equation

$$M_S = M_A \begin{bmatrix} A & B \\ B & -A \end{bmatrix}.$$

Equivalently, it is given by the system of Beltrami equations

$$\overline{D}F = DF\mu. \tag{2.10}$$

Let $\| \mu \|_\infty = \sup \| \mu \|$. If $\| \mu \|_\infty \leq k < 1$ then F is called *K–symplectic quasiconformal* where $K = \frac{1-k}{1+k}$.

2.5. **The boundary. Contact structure.** The boundary $\partial\mathbf{H}^2_{\mathbb{C}}$, i.e. the paraboloid $2\Re(z_1)+|z_2|^2=0$ is identified to the one point compactification of the Heisenberg group $\overline{\mathfrak{H}}$ via the map

$$\partial\mathbf{H}^2_{\mathbb{C}} \ni (z_1,z_2)\mapsto [\zeta,t]\in\overline{\mathfrak{H}},\quad \zeta=\frac{1}{\sqrt{2}}z_2,\quad t=\Im(z_1).$$

The contact structure of $\partial\mathbf{H}^2_{\mathbb{C}}$ is obtained as a strongly pseudoconvex CR structure as follows. Consider the 1–form $\omega=-\frac{1}{2}\mathbb{J}d\rho$, where $\mathbb{J}$ is the standard complex structure restricted to the boundary. Explicitly, in Heisenberg coordinates $\zeta=x+iy,t$,

$$\omega=dt+2(xdy-ydx).$$

Consider the vector fields

$$X=\frac{\partial}{\partial x}+2y\frac{\partial}{\partial t},\quad Y=\frac{\partial}{\partial y}-2x\frac{\partial}{\partial t},\quad T=\frac{\partial}{\partial t}$$

and also the complex fields

$$Z=\frac{1}{2}(X-iY)=\frac{\partial}{\partial\zeta}+i\overline{\zeta}\frac{\partial}{\partial t}\quad \overline{Z}=\frac{1}{2}(X+iY)=\frac{\partial}{\partial\overline{\zeta}}-i\zeta\frac{\partial}{\partial t}.$$

The tangent space to $\partial\mathbf{H}^2_{\mathbb{C}}$ is spanned by X,Y,T and the kernel of ω, else called the horizontal space H of $\partial\mathbf{H}^2_{\mathbb{C}}$ is spanned by the vector fields X,Y. The vector field T is called the Reeb vector field and satisfies $[X,Y]=-4T$, $i(T)d\omega=0$ and $\omega(T)=1$. The Levi form $d\omega$ defines a positive definite hermitian form on H since $d\omega(X,\mathbb{J}Y)>0$. Thus the CR-structure ω is strictly pseudoconvex and thus contact. The volume form on $\partial\mathbf{H}^2_{\mathbb{C}}$ is $\omega\wedge d\omega=4dx\wedge dy\wedge dt$.

2.5.1. *Contactomorphisms and quasiconformal mappings.* A *contactomorphism* of $\mathbf{H}^2_{\mathbb{C}}$ is a self mapping f enjoying the following properties.

(1) $f:\mathbf{H}^2_{\mathbb{C}}\to\mathbf{H}^2_{\mathbb{C}}$ is a diffeomorphism and
(2) its continuation to the boundary, denoted again by f, satisfies $f_*(\mathrm{H})=\mathrm{H}$. This is equivalent to say that $f^*\omega=\tau\omega$ for some function τ.

We shall use the following definition for quasiconformal mappings, see Theorem 7 of [13] or Theorem C of [12] and the reader should mind the difference of normalisation of the Siegel domain which, nevertheless, plays no important role.

Definition 2.3. Assume that $f=(f_1,f_2)$, $2\Re(f_1)+|f_2|^2=0$, is a $\mathcal{C}^2-$diffeomorphism. Then it is $K-$quasiconformal if and only if there exists a complex valued function μ with $\|\mu\|_\infty\leq\frac{K-1}{K+1}$ such that f_i satisfy the equations

$$\overline{Z}f_i=\mu Zf_i,\quad i=1,2. \tag{2.11}$$

2.5.2. *Symplectomorphisms vs. contactomorphisms.* The following Proposition summarises some well known facts, see [10].

Proposition 2.4. *(1) A (quasiconformal) symplectomorphism f of the complex hyperbolic plane extends to a (quasiconformal) contactomorphism in the boundary.*

(2) A $\mathcal{C}^2$ vector field V generates a one parameter group f_s of contactomorphisms if and only if is of the form

$$V=-\frac{1}{4}\left((Yp)X-(Xp)Y\right)+pT \tag{2.12}$$

for some real valued function p.

(3) If moreover $|ZZp| \le k$ *then* f_s *is* $K-$*quasiconformal with*

$$\frac{1}{2}\left(K+\frac{1}{K}\right) \le e^{\sqrt{2}k|s|}$$

(4) A smooth one parameter family f_t *of (quasiconformal) contactomorphisms,* $t \in [0,1]$, $f_0 = id.$ *extends to a one parameter family of (quasiconformal) symplectomorphisms in the interior.*

3. Packs

Packs are the counterpart of bisectors: in general, a pack is real analytic 3-dimensional submanifold of complex hyperbolic plane which is naturally foliated by Lagrangian planes. The simplest case of packs (the *flat* packs in our terminology) was first presented by Will; see [17]. The general definition we give below may be found in [15]. We have pointed out in section 2.3.2 that bisectors are characterised only by the endpoints of their spine and subsequently, $\mathrm{SU}(2,1)$ acts transitively on the set of bisectors. In the case of packs, this is not true in general. In Section 3.1 we give a necessary and sufficient condition under which two packs can be mapped onto each other via an element of $\mathrm{SU}(2,1)$. This will enable us to describe the CR-geometry of packs in Section 3.2. We first recall the definition of a pack.

Definition 3.1. Let R_1, R_2 be disjoint Lagrangian planes, I_i, $i = 1,2$ be inversions on R_i and $C = I_2 \circ I_1$ be the loxodromic element of $\mathrm{SU}(2,1)$ induced by I_1 and I_2. The set

$$P(R_1;C) = \bigcup_{x\in\mathbb{R}} C^{x/2}(R_1)$$

is called the *pack associated to the loxodromic element* C. The real axis γ of C (oriented from the repulsive fixed point to the attractive fixed point of C) is called the *real spine* of P and the complex axis $L_{\mathbb{C}}$ of P is called the *complex spine* of P. The complex length $\lambda = l + i\theta \in S$ of C is called the *complex length* of P.

A pack $P(R_1;C)$ is invariant under some obvious isometries. These are described in the next Lemma.

Lemma 3.2. *Let* $P(R_1;C)$ *be a pack. Then the following hold.*

(1) If $A \in \mathrm{SU}(2,1)$ *then*

$$A(P(R_1;C)) = P(A(R_1); ACA^{-1}).$$

(2) $C^x(P(R_1;C)) = P(C^x(R_1);C) = P(R_1;C)$ *for all fixed* $x \in \mathbb{R}$.

(3) Let I_1 *be inversion on the Lagrangian plane* R_1*. Then* $I_1(P(R_1;C)) = P(R_1;C)$*. In particular, let*

$$P^+ = \bigcup_{x\ge 0} C^{x/2}(R_1), \quad P^- = \bigcup_{x\le 0} C^{x/2}(R_1).$$

Then

$$I_1(P^+) = P^-, \quad I_1(P^-) = P^+.$$

By Lemma 3.2 we obtain that a pack $P(R_1;C)$ depends only on the loxodromic element C and not on R_1. Thus from now on we shall write $P(C)$ instead of $P(R_1;C)$.

Corollary 3.3. *Let $P = P(C)$ be a pack. We may normalise so that*

$$P = \bigcup_{x \in \mathbb{R}} C^{x/2}(R_1')$$

where R_1' is a Lagrangian plane passing through the intersection point of $L_{\mathbb{C}}$ and its orthogonal $L_{\mathbb{C}}^{\perp}$.

Proof. Let z_0 be the intersection point of $L_{\mathbb{C}}$ and $L_{\mathbb{C}}^{\perp}$. Let γ_0 be the geodesic in $L_{\mathbb{C}}$ which passes from z_0 and is the intersection of $L_{\mathbb{C}}$ and $C^{x_0/2}(R_1)$ for some $x_0 \in \mathbb{R}$. Let $R_1' = R_{x_0} = C^{x_0/2}(R_1)$. Our claim then follows from properties (1) and (2) of Lemma 3.2. □

3.1. **The action of the isometry group.** As we have already mantioned, and in contrast to the case of bisectors, the isometry group of complex hyperbolic plane does not in general act transitively on packs. We shall prove in this section that transitive action is controlled by a certain number associated to each pack, called the *curl factor*.

Definition 3.4. Let $P = P(C)$ be a pack, $C = QE(\lambda)Q^{-1}$ where e^{λ} is the attractive eigenvalue of C.

(1) The *curl factor* $\kappa = \kappa(P)$ *of the pack* P is defined to be $\kappa = \theta/l = \tan(\arg(\lambda))$.
(2) The pack P shall be called *flat* if its curl factor is 0.

It is obvious that a pack $P = P(C)$ is flat if and only if C is conjugate to an element of $\mathrm{SO}(2,1)$. We now proceed to the main Theorem of this section.

Theorem 3.5. *Two packs P_1 and P_2 with curl factors κ_1 and κ_1 respectively are isometric if and only if $\kappa_1 = \kappa_2$.*

Lemma 3.6. *Let $P = P(C)$ be a pack with complex length λ. Then P is isometric to the pack $P(E(\lambda))$.*

Proof. We may suppose that P is normalised as in Corollary 3.3 and let also $C = QE(\lambda)Q^{-1}$. Then Q^{-1} maps $P(C)$ to $P(E(\lambda))$. By applying a rotation around the real axis of $E(\lambda)$, we may also suppose that

$$P(E(\lambda)) = \bigcup_{x \in \mathbb{R}} E^{x/2}(\lambda)(R_{\mathbb{R}}),$$

where $R_{\mathbb{R}}$ is the standard real Lagrangian plane as in Equation 2.8. □

Lemma 3.7. *Suppose $P_j = P(E(\lambda_j))$, $\lambda_j = l_j + i\theta_j \in S$, $j = 1, 2$. Let also $\kappa_j = \theta_j/l_j$ be the curl factor of P_j, $j = 1, 2$. Then P_1 and P_2 are isometric if and only if $\kappa_1 = \kappa_2$.*

Proof. Suppose first that P_1 is isometric to P_2. Since an isometry preserves complex lengths, then we must have $\lambda_1 = \lambda_2$ and thus $\kappa_1 = \kappa_2$.

Conversely, suppose that $\kappa_1 = \kappa_2 = \kappa$. Then we may write

$$\begin{aligned} P(E(\lambda_j)) &= \bigcup_{x \in \mathbb{R}} E^{x/2}(\lambda_j)(R_{\mathbb{R}}) \\ &= \bigcup_{x \in \mathbb{R}} E^{l_j x/2}(1 + i\kappa)(R_{\mathbb{R}}) \end{aligned}$$

Let now $x_1 \in \mathbb{R}$ and $E^{l_1 x_1/2}(1 + i\kappa)(R_{\mathbb{R}})$ be a slice of P_1. Then there exists a $x_2 \in \mathbb{R}$, $x_2 = l_1 x_1/l_2$ such that $E^{l_1 x_1/2}(1+i\kappa)(R_{\mathbb{R}}) = E^{l_2 x_2/2}(1+i\kappa)(R_{\mathbb{R}})$ is also a slice of P_2. Analogously we may show that any slice of P_2 is also a slice of P_1. Therefore $P_1 = P_2$ and the proof is complete. □

Proof of Theorem 3.5. By Lemma 3.6 we may restrict ourselves to the case where $P_j = P(E(\lambda_j))$, $j = 1, 2$. Then our claim follows directly from Lemma 3.7. □

Corollary 3.8. *Any two flat packs are isometric.*

From the proof of Lemma 3.7 it follows that any pack with complex length λ and curl factor κ is isometric to the pack

$$P_\kappa = P(E(1+i\kappa)) = \bigcup_{xi\in\mathbb{R}} E^{\xi/2}(1+i\kappa)(R_\mathbb{R}). \tag{3.1}$$

Definition 3.9. The pack P_κ is called the *standard pack with curl factor* κ.

The standard pack P_κ is the prototype of a pack in the subsequent discussion. Therefore we would like to have a neat normalisation for P_κ and we indeed obtain one in the next section.

3.1.1. *Represantation of the standard pack.* In this section we represent the standard pack P_κ in a manner which identifies P_κ to an infinite cylinder in $\mathbb{R}^3$. To do so, we use the normalisation of the standard Lagrangian plane $R_\mathbb{R}$ given in Equation 2.8.

Proposition 3.10. *Let P_κ be the standard pack with curl factor κ. Then*

$$P_\kappa = \left\{\left(-e^{\xi+i\psi}, ire^{\xi/2+i(\psi-3\kappa\xi)/2}\sqrt{2\cos(\psi)}\right) \in \mathbb{C}^2; \quad (\xi,\psi,r) \in \mathbb{R}\times I_\psi \times I_r\right\} \tag{3.2}$$

where

$$I_\psi = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad I_r = (-1,1).$$

Proof. We first write

$$P_\kappa = \bigcup_{\xi\in\mathbb{R}} E^{\xi/2}(1+i\kappa)(R_\mathbb{R})$$

and recall from Equation 2.8 that the standard real Lagrangian plane $R_\mathbb{R}$ may be written as

$$R_\mathbb{R} = \left\{\mathbf{r} = \begin{bmatrix} -e^{i\psi} \\ ir\sqrt{2\cos(\psi)}e^{i\psi/2} \\ 1 \end{bmatrix} \in V_- : (\psi, r) \in I_\psi \times I_r \right\}.$$

Applying $E^{\xi/2}(1+i\kappa)$ to each $\mathbf{r} \in R_\mathbb{R}$ we have

$$\begin{aligned} E^{\xi/2}(1+i\kappa)(\mathbf{r}) &= \begin{bmatrix} e^{(1+i\kappa)\xi/2} & 0 & 0 \\ 0 & e^{-i\kappa\xi} & 0 \\ 0 & 0 & e^{(-1+i\kappa)\xi/2} \end{bmatrix} \begin{bmatrix} -e^{i\psi} \\ ir\sqrt{2\cos(\psi)}e^{i\psi/2} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{(1+i\kappa)\xi/2+i\psi} \\ ire^{i(\psi/2-\kappa\xi)}\sqrt{2\cos(\psi)} \\ e^{(-1+i\kappa)\xi/2} \end{bmatrix} \\ &\sim \begin{bmatrix} -e^{\xi+i\psi} \\ ire^{\xi/2+i(\psi-3\kappa\xi)/2}\sqrt{2\cos(\psi)} \\ 1 \end{bmatrix}. \end{aligned}$$

This completes the proof. □

3.2. **CR Geometry.** The fact that by definition a pack is a 3-dimensional real analytic hypersurface of $\mathbf{H}^2_{\mathbb{C}}$ foliated by Lagrangian planes automatically gives rise to questions about its CR geometry. In the case of a bisector, it is naturally foliated by complex lines and also, it may as well be seen as the union of the Lagrangian planes which all meet at the real spine of the bisector. In the case of a pack, by definition it is foliated by Lagrangian planes. Therefore it remains to check if is also foliated by 1-complex dimensional objects, a problem which is naturally associated to the study of its CR geometry.

Since packs are characterised by their curl factor, it is sufficient to study the CR geometry of the standard pack P_κ with curl factor κ.

Corollary 3.11. *The standard pack P_κ with curl factor κ is defined as a (singular) differentiable hypersurface of $\mathbf{H}^2_{\mathbb{C}}$ by the Equation*

$$\mathfrak{p}_\kappa(z_1,z_2)=\arg\left(\frac{z_2^2}{z_1}\right)+3\kappa\log(|z_1|)=0 \mod (2\pi) \tag{3.3}$$

Proof. If $(z_1,z_2)\in P_\kappa$ then from Equation 3.2 we have

$$z_1=-e^{\xi+i\psi},\quad z_2=ire^{\xi/2+i(\psi-3\kappa\xi)/2}\sqrt{2\cos(\psi)}$$

for some $(\xi,\psi,r)\in\mathbb{R}\times I_\psi\times I_r$. Now,

$$\frac{z_2^2}{z_1}=2r^2e^{-3i\kappa\xi}\cos(\psi).$$

Therefore,

$$\arg\left(\frac{z_2^2}{z_1}\right)=-3\kappa\log(|z_1|)\mod (2\pi).$$

Finally, it is clear that $\mathfrak{p}_\kappa$ is differentiable except at points of the complex spine $L_0=\{(z_1,z_2):z_2=0\}$. □

Let $z\in P_\kappa-L_0$ be an arbitrary point. The horizontal space H_z of $P_\kappa-L_0$ at z is defined to be the intersection of $\mathrm{T}_z(\mathbf{H}^2_{\mathbb{C}})$ and $\mathbb{J}\mathrm{T}_z(\mathbf{H}^2_{\mathbb{C}})$ where $\mathbb{J}$ is the natural complex operator of $\mathbf{H}^2_{\mathbb{C}}$ with its action restricted at points of $P_\kappa-L_0$. The CR structure of $P_\kappa-L_0$ is the distribution

$$\mathcal{E}(z)=\{z,\mathrm{H}_z\}=\ker(d^cF_\kappa)_z.$$

In order to determine $\mathcal{E}$ we equivalently determine

$$\mathcal{E}^{(1,0)}(z)=\{z,\mathrm{H}_z^{(1,0)}\}=\ker(\partial\mathfrak{p}_\kappa)_z.$$

We have

$$\begin{aligned}\partial\mathfrak{p}_\kappa &= \frac{\partial\mathfrak{p}_\kappa}{\partial z_1}dz_1+\frac{\partial\mathfrak{p}_\kappa}{\partial z_2}dz_2\\ &= -\frac{3\kappa+i}{2z_1}dz_1+\frac{i}{z_2}dz_2.\end{aligned}$$

Thus, $\mathcal{E}^{(1,0)}$ is generated by the holomorphic vector field

$$Z=2iz_1\frac{\partial}{\partial z_1}+(3\kappa+i)z_2\frac{\partial}{\partial z_2}.$$

A CR structure is completely integrable if and only if for each $Z,W\in\mathcal{E}^{(1,0)}$ the Lie bracket $[Z,\overline{W}]$ is also in $\mathcal{E}^{(1,0)}$. In our case this is true since Z is holomorphic and thus $[Z,\overline{Z}]=0$. Therefore,

$P_\kappa - L_0$ is foliated by 1-complex dimensional submanifolds which are determined by solving the differential equation

$$\frac{3\kappa+i}{2z_1}dz_1 = \frac{i}{z_2}dz_2.$$

Thus the leaves are given by

$$z_2 = c z_1^{\frac{1-3\kappa i}{2}},$$

where c is a complex constant depending on the point z. The previous discussion is summarised to the next Proposition which describes the decomposition of packs into complex submanifolds.

Proposition 3.12. *Let P be a pack with curl factor κ. Then P admits a singular codimension 1 foliation which is such that*

1. *its singular leaf is the complex spine of P and*
2. *each non singular leaf is biholomorphic to the Riemann surface*

$$w = z^{\frac{1-3\kappa i}{2}}.$$

4. Cylindrical Model

Towards the further study of the geometry of packs, and motivated by the representation of the standard pack given in section 3.1.1, we introduce here a set of coordinates for $\mathbf{H}^2_{\mathbb{C}}$ which arise naturally from our previous discussion. Moreover, these coordinates may as well be considered as the dual to the cylindrical coordinates of Goldman and Corlette, see Section 5.5 of [6]. In the latter, the standard bisector B_0 defined in Equation 2.9 is parametrised by a simple equation, in fact it is the zero image of one of the coordinate functions. In what follows we show that there exists an analogous result in the case of packs. The construction we present below will provide a model for complex hyperbolic plane which is proved to be quite handy for our purposes.

4.1. **Cylindrical coordinates.** Cylindrical coordinates for complex hyperbolic plane are defined by the following Lemma.

Lemma 4.1. *Let*

$$D = \{z \in \mathbb{C} :\ |z| < 1\}, \quad I_\psi = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We consider the following subdomain of $\mathbb{C}^2$:

$$\mathfrak{C} = \{(w,z) : w = \xi + i\psi \in \mathbb{R} \times I_\psi, z = u + iv \in D\}.$$

The mapping $\Xi : \mathfrak{C} \to \mathbb{C}^2$ defined by

$$\Xi(\xi,\psi,u,v) = \left(-e^{\xi+i\psi}, (u+iv)\sqrt{2\cos(\psi)}e^{(\xi+i\psi)/2}\right) \tag{4.1}$$

is a differentiable one–to–one transformation of $\mathfrak{C}$ onto the Siegel domain of $\mathbb{C}^2$ with a differentiable inverse. Its inverse Ξ^{-1} is also differentiable. For each $(z_1,z_2) \in \mathbf{H}^2_{\mathbb{C}}$ we have

$$\Xi^{-1}(z_1,z_2) = (w,z) = \left(\log(z_1) - i\pi, \frac{iz_2 e^{-i\arg(z_1)/2}}{\sqrt{-2\Re(z_1)}}\right) \tag{4.2}$$

where here $\log(z_1)$ is the branch of the logarithm for which $\arg(z_1) \in (\pi/2, 3\pi/2)$.

Proof. To prove that Ξ is a one–to–one mapping onto the Siegel domain is trivial and we leave the details to the reader. To prove differentiability we write first Ξ in real terms as

$$\Xi(\xi,\psi,u,v) = (x_1,y_1,x_2,y_2)$$

where

$$\begin{aligned} x_1 &= -e^{\xi}\cos(\psi),\\ y_1 &= -e^{\xi}\sin(\psi),\\ x_2 &= e^{\xi/2}\sqrt{2\cos(\psi)}(u\cos(\psi/2)-v\sin(\psi/2)),\\ y_2 &= e^{\xi/2}\sqrt{2\cos(\psi)}(v\cos(\psi/2)+u\sin(\psi/2)). \end{aligned}$$

The Jacobian determinant $J_\Xi = \det(d\Xi)$ of Ξ is then

$$\begin{aligned} J_\Xi &= \left|\frac{\partial(x_1,y_1,x_2,y_2)}{\partial(\xi,\psi,u,v)}\right|\\ &= \begin{vmatrix} x_1 & -y_1 & 0 & 0\\ y_1 & x_1 & 0 & 0\\ x_2/2 & * & \sqrt{-2x_1}\cos(\psi/2) & -\sqrt{-2x_1}\sin(\psi/2)\\ y_2/2 & * & \sqrt{-2x_1}\sin(\psi/2) & \sqrt{-2x_1}\cos(\psi/2) \end{vmatrix}\\ &= (x_1^2+y_1^2)\cdot(-2x_1)\\ &= 2e^{3\xi}\cos(\psi) \end{aligned}$$

which is clearly positive everywhere in $\mathfrak{C}$.

Finally we prove Equation 4.2. To do so, we fix a $(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ and we may suppose that $\arg(z_1) \in (\pi/2, 3\pi/2)$ and $\arg(z_2) \in (-\pi, \pi]$. Then from $z_1 = -e^{\xi+i\psi}$ we have

$$\xi = \log(|z_1|), \quad \psi = \arg(z_1) - \pi.$$

Thus $z_2 = (u + iv)\sqrt{2\cos(\psi)}e^{(\xi+i\psi)/2}$ yields

$$u + iv = \frac{iz_2 e^{-i\arg(z_1)/2}}{\sqrt{-2\Re(z_1)}}.$$

□

From now on we identify complex hyperbolic plane with $\mathfrak{C}$. The domain

$$\mathfrak{C} = \mathbb{R} \times I_\psi \times D$$

resembles the *cylindrical model* for complex hyperbolic plane. It is possible to express all standard features of complex hyperbolic plane in terms of cylindrical coordinates $w = \xi + i\psi, z = u + iv$. However, we prefer for our purposes to use "polar" coordinates (ξ, ψ, r, η) where r, η are defined by the relation $z = ire^{-3i\eta/2}$, with $r \in (-1, 1)$ and $\eta \in \left(-\frac{2\pi}{3}, 0\right)$. The reason why we did not define cylindrical coordinates to be ξ, ψ, r, η in the first place, is to avoid in the proof of Lemma 4.1 the usual polar coordinates singularity at points where $r = 0$ (Observe that the $\Xi-$image of these points are the complex line L_0). Nevertheless, this singularity does not affect the whole picture and moreover, in terms of coordinates ξ, ψ, r, η, the expressions of the geometrical features of complex hyperbolic plane as well as the expression for the defining function of the standard pack P_κ are much more simpler, as we explain below.

Now, Equations 2.9 and 3.3 together with Lemma 4.1 yield the following.

Corollary 4.2. *In the cylindrical model of complex hyperbolic plane:*

(1) The standard pack P_κ with curl factor κ is defined the equation $\eta = \kappa\xi \mod (2\pi/3)$.
(2) The standard bisector B_0 with complex spine L_0 is defined by the equation $\psi = 0$.
(3) The intersection $S_0 = B_0 \cap P_0$ is the Lagrangian plane $2x_1 + x_2^2 < 0,\ x_1 < 0,\ x_2 \in \mathbb{R}$.

4.2. **Complex structure.** We are particularly interested in the expression of the Riemannian as well as of the symplectic structure of complex hyperbolic plane in cylindrical coordinates. For this, it is necessary to start by describing the action of the standard complex structure $\mathbb{J}$ in terms of ξ, ψ, r, η.

Proposition 4.3. *The natural complex operator* $\mathbb{J}$ *of* $\mathbf{H}^2_{\mathbb{C}}$ *is expressed in cylindrical coordinates by the following matrix equations.*

$$\mathbb{J}\begin{pmatrix} d\xi \\ d\psi \\ dr \\ d\eta \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -\frac{r}{2}\tan(\psi) & 0 & 0 & -\frac{3r}{2} \\ 0 & -\frac{\tan(\psi)}{3} & \frac{2}{3r} & 0 \end{pmatrix}\begin{pmatrix} d\xi \\ d\psi \\ dr \\ d\eta \end{pmatrix}, \tag{4.3}$$

which describes the action of $\mathbb{J}$ *in the cotangent space and*

$$\mathbb{J}\begin{pmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\psi} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{r}{2}\tan(\psi) & 0 \\ -1 & 0 & 0 & \frac{\tan(\psi)}{3} \\ 0 & 0 & 0 & -\frac{2}{3r} \\ 0 & 0 & \frac{3r}{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\psi} \\ \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\eta} \end{pmatrix} \tag{4.4}$$

which describes the action of $\mathbb{J}$ *in the tangent space of* $\mathbf{H}^2_{\mathbb{C}}$.

Proof. By taking logarithms in equations

$$z_1 = -e^{\xi+i\psi}, \quad z_2 = ir\sqrt{2\cos(\psi)}e^{\xi/2+i(\psi-3\eta)/2}$$

we obtain

$$\log(z_1) = \xi + i\psi + i\pi, \tag{4.5}$$

$$\log(z_2) = \frac{1}{2}\left(\xi + \log(2\cos(\psi))\right) + \log(r) + \frac{i}{2}(\psi - 3\eta + \pi). \tag{4.6}$$

Thus

$$\frac{dz_1}{z_1} = d\xi + id\psi, \tag{4.7}$$

$$\frac{dz_2}{z_2} = \frac{1}{2}\left(d\xi - \tan(\psi)d\psi\right) + \frac{dr}{r} + \frac{i}{2}(d\psi - 3d\eta). \tag{4.8}$$

By applying $\mathbb{J}$ to both parts of 4.7 and 4.8 and then using the relations $\mathbb{J}(dz_i) = -idz_i$ we obtain equation 4.3. Equation 4.4 follows immediately. □

5. Transitive Action of the Symplectic Group

In this section we prove our first main

Theorem 5.1. *The group of symplectomorphisms* $\mathbf{Sp}(\mathbf{H}^2_{\mathbb{C}})$ *acts transitively in packs.*

The main key to the proof is the construction of a symplectomorphism F_κ of $\mathfrak{C} \equiv \mathbf{H}^2_{\mathbb{C}}$ which maps the standard flat pack P_0 onto the standard pack P_κ with curl factor κ. This symplectomorphism admits a simple expression in cylindrical cordinates (ξ, ψ, r, η). Thus we start by expressing the symplectic form Ω of complex hyperbolic plane in these coordinates. To do so, we identify the defining function ρ of $\mathbf{H}^2_{\mathbb{C}}$ as in 2.1 with the function $\rho \circ \Xi$. In this manner, $\rho : \mathfrak{C} \to \mathbb{R}$ is given by

$$\rho(\xi, \psi, r, \eta) = 2e^{\xi}\cos(\psi)(1 - r^2). \tag{5.1}$$

Proposition 5.2. *In the cylindrical model of complex hyperbolic plane, the Bergman-Kähler symplectic form Ω is given by*

$$\Omega = -\frac{1}{(1-r^2)^2}\left((1-r^2)(1+\tan^2(\psi))d\xi\wedge d\psi + 2r\tan(\psi)d\xi\wedge dr + 6rd\eta\wedge dr\right). \tag{5.2}$$

Proof. The Bergman-Kähler symplectic form is $\Omega = dd^c\log(\rho)$. We use equations 4.3 to obtain

$$\begin{aligned} d^c\log(\rho) = \mathbb{J}d\log(\rho) &= \mathbb{J}d\xi - \tan(\psi)\mathbb{J}d\psi - \frac{2r}{1-r^2}\mathbb{J}dr \\ &= d\psi + \frac{1}{1-r^2}(\tan(\psi)d\xi + 3r^2 d\eta). \end{aligned}$$

Formula 5.2 is then obtained by applying the differential operator d in both sides of the above equation. □

Lemma 5.3. *Let P_κ be the standard pack with curl factor κ and let also $F_\kappa : \mathfrak{C}\to\mathbb{C}^2$ where*

$$F_\kappa(\xi,\psi,r,\eta) = (\xi, \arctan(\tan(\psi)-3\kappa), r, \eta+\kappa\xi \bmod (2\pi/3)). \tag{5.3}$$

The mapping F_κ is a diffeomorphism of $\mathfrak{C}$ and also satisfies the following.

(1) $F_\kappa^*\Omega = \Omega$ and
(2) F_κ maps the standard flat pack P_0 onto P_κ.

Proof. Writing Equation 5.3 in the equivalent form

$$F_\kappa(\xi,\psi,u,v) = (\xi, \arctan(\tan(\psi)-3\kappa), u\cos(3\kappa\xi)+v\sin(3\kappa\xi), v\cos(3\kappa\xi)-u\sin(3\kappa\xi)) \tag{5.4}$$

we may verify easily that F_κ is a one–to–one and onto self mapping of $\mathfrak{C}$. Next, F_κ is clearly differentiable and its Jacobian determinant is

$$\begin{aligned} \det(dF_\kappa) &= \left|\frac{\partial F_\kappa(\xi,\psi,u,v)}{\partial(\xi,\psi,u,v)}\right| \\ &= \frac{1+\tan^2(\psi)}{1+(\tan(\psi)-3\kappa)^2} \end{aligned}$$

which is positive everywhere in $\mathfrak{C}$. Thus F_κ is a diffeomorphism of $\mathfrak{C}$.

We now use Equation 5.3 and Proposition 5.2 to obtain $F_\kappa^*\Omega = \Omega$. Our second assertion follows from

$$F_\kappa(\xi,\psi,r,0) = (\xi, \arctan(\tan(\psi)-3\kappa), r, \kappa\xi \bmod (2\pi/3))$$

which implies $F_\kappa(P_0) = P_\kappa$ since the defining function of P_κ is independent of ψ. The proof is thus complete. □

Proof of Theorem 5.1. Let P and P' be any two packs with curl factors κ and κ' respectively. We have to show that there exists a symplectomorphism $F : \mathbf{H}^2_{\mathbb{C}} \to \mathbf{H}^2_{\mathbb{C}}$ such that $F(P) = P'$. Now we may isometrically map P and P' respectively to the standard packs P_κ and P'_κ. Thus, we only have to show the existence of a symplectomorphism $F : \mathbf{H}^2_{\mathbb{C}} \to \mathbf{H}^2_{\mathbb{C}}$ such that $F(P_\kappa) = P_{\kappa'}$. Let P_0 be the standard flat pack. By Lemma 5.3 there exist symplectomorphisms $F_\kappa, F_{\kappa'}$ mapping P_0 respectively to P_κ and $P_{\kappa'}$. Thus the desired symplectomorphism is $F = F_{\kappa'}\circ F_\kappa^{-1}$ and this concludes the proof. □.

5.1. $\kappa-$**Bisectors.** Let B_0 be the standard bisector and consider its F_κ image $B_\kappa = F_\kappa(B_0)$. This is a new 3–submanifold of complex hyperbolic plane, with defining function given in cylindrical coordinates by $\psi = \arctan(-3\kappa)$. Now F_κ preserves the complex spine L_0 therefore L_0 is contained in B_κ. The real spine $s_0 = (-\infty, 0)$ is mapped onto the Euclidean straight line (that is a horocycle) on L_0 given by $s_\kappa : \psi = \arctan(-3\kappa),\ r = 0,\ \eta = 0$ or, in standard coordinates, $s_\kappa : \arg(z_1) = \pi - \arctan(3\kappa),\ z_2 = 0$. Hence the hyperbolic distance $\delta(s_\kappa, s_0)$ is given by

$$\cosh(\delta(s_\kappa, s_0)) = -\frac{1}{\cos(\theta_0)},\ \theta_0 = \pi - \arctan(3\kappa),$$

(see for instance formulae 7.20.3 of [2] but mind the opposite sign due to the different normalisation of the hyperbolic plane).

Since F_κ is a symplectomorphism, there is a meridianal decomposition of B_κ by Lagrangian planes meeting at s_κ, which is exactly the $F_\kappa-$image of the meridianal decomposition of B_0. Let now Π_{L_0} be the projection in L_0. In standard coordinates $\Pi_{L_0}(z_1, z_2) = (z_1, 0)$ and thus in cylindrical coordinates

$$\Pi_{L_0}(\xi, \psi, u, v) = (\xi, \psi, 0, 0).$$

Thus F_κ commutes with Π_{L_0} and therefore

$$\begin{aligned} B_\kappa = F_\kappa(B_0) &= \bigcup_{x \in s_0} F_\kappa(\Pi_{L_0}^{-1}(x)) \\ &= \bigcup_{x \in s_0} \Pi_{L_0}^{-1}(F_\kappa(x)) \\ &= \bigcup_{y \in s_\kappa} \Pi_{L_0}^{-1}(y) \end{aligned}$$

and this resembles the foliation of B_κ by complex lines. If we carry out the same discussion but instead of F_κ we use its inverse $F_\kappa^{-1} = F_{-\kappa}$ then we end up with

$$B_{-\kappa} = \bigcup_{y \in s_{-\kappa}} \Pi_{L_0}^{-1}(y)$$

where $s_{-\kappa}$ is the Euclidean straight line $s_{-\kappa} : \arg(z_1) = \pi + \arctan(3\kappa),\ z_2 = 0$. Observe that if $\delta(s_{-\kappa}, s_0)$ is the hyperbolic distance of $s_{-\kappa}$ and s_0 then

$$\cosh(\delta(s_{-\kappa}, s_0)) = \cosh(\delta(s_\kappa, s_0)) = \frac{1}{\cos(\arctan(3\kappa))}.$$

The above discussion lead us naturally to the following definition.

Definition 5.4. Let $B = B(s)$ be a bisector with real spine s and complex spine $L_{\mathbb{C}}$. Let also κ be a real number. The two 3–hypersurfaces defined by

$$B(s_{\pm\kappa}) = \left\{ z \in \mathbf{H}^2_{\mathbb{C}} : \cosh(\delta(z, s)) = \frac{1}{\cos(\arctan(3\kappa))} \right\},$$

are called the $\kappa-$*bisectors associated to* B with horocyclic spines $s_{\pm\kappa}$ given by

$$B(s_{\pm\kappa}) \cap L_{\mathbb{C}} = \left\{ z \in L_{\mathbb{C}} : \cosh(\delta(z, s)) = \frac{1}{\cos(\arctan(3\kappa))} \right\}.$$

From our previous discussion we obtain the following Proposition which summarises some of the basic properties enjoyed by $\kappa-$bisectors.

Proposition 5.5. *Let $B = B(s)$ be a bisector with real spine s and complex spine $L_{\mathbb{C}}$ and consider its asociated $\kappa-$bisectors for some $\kappa \in \mathbb{R}$. Let also $\Pi_{L_{\mathbb{C}}} : \mathbf{H}^2_{\mathbb{C}} \to L_{\mathbb{C}}$ be the orthogonal projection in $L_{\mathbb{C}}$. Then the following hold.*

(1) There exist symplectomorphisms of complex hyperbolic plane mapping each $\kappa-$bisector onto the bisector $B(s)$.

(2) (Slice decomposition). $B(s_{\pm\kappa})$ are foliated by complex lines:

$$B(s_{\pm\kappa}) = \bigcup_{x \in s_{\pm\kappa}} \Pi^{-1}_{L_{\mathbb{C}}}(x).$$

(3) (Meridianal decomposition). $B(s_{\pm\kappa})$ are the union of all Lagrangian planes meeting at their horocyclic spines.

Proof. To prove (1) consider an isometry G mapping B to the standard bisector B_0 and let also $F_{\pm\kappa}$ be as in Lemma 5.3. Then $F^{\pm} = G^{-1} \circ F_{\pm\kappa} \circ G$ is a symplectomorphism of $\mathbf{H}^2_{\mathbb{C}}$ and it is easy to see using the definition of $\kappa-$bisectors that $F^{\pm}(B(s_{\pm\kappa})) = B(s)$. Now (2) and (3) follow immediately from (1) and the discussion in the beginning of this Section. □

6. Quasiconformal Deformation

In this section we construct a family of quasiconformal mappings of the Heisenberg group $\mathcal{H}$ associated to a family of packs. We are doing so with the aid of the symplectomorphism F_κ defined in the previous section. The idea is the following. Suppose that $\{P_\tau\}$, $\tau \in [0,1]$ is a family of packs (see 6.2 below). For simlicity we may assume that all P_τ are standard. Based on Lemma 5.3 we construct a globally defined Hamiltonian time dependent vector field which generates a flow of symplectomorphisms $\phi s, \tau$ such that $\phi_{0,\tau}(P_0) = P_\tau$. This vector field can be extended to the closure of complex hyperbolic plane and there, it generates a flow of quasiconformal contactomorphisms. By Proposition 2.4 (4), this flow is also a flow of quasiconformal symplectomorphisms in the interior. Again, we find very convenient to use cylindrical coordinates for our calculations, as these are extended to $\partial\mathbf{H}^2_{\mathbb{C}}$.

6.1. **Cylindrical model: Extension to the boundary.** Cylindrical coordinates extend to the boundary in a natural manner.

Lemma 6.1. *Let*

$$\overline{D} = \{z \in \mathbb{C} : \ |z| \leq 1\}, \quad \overline{I_\psi} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

We consider the domain

$$\overline{\mathfrak{C}} = \left\{(w,z) : \ w = \xi + i\psi \in \mathbb{R} \times \overline{I_\psi}, z = u + iv \in \overline{D}\right\}.$$

Then cylindrical coordinates are naturally extended to the closure of complex hyperbolic plane in the sense that the boundary $\partial\mathbf{H}^2_{\mathbb{C}}$ of $\mathbf{H}^2_{\mathbb{C}}$ is identified to $\mathbb{R} \times \overline{I_\psi} \times \ \partial D$ and the mapping $\widetilde{\Xi} : \partial(\mathfrak{C}) \to \partial\mathbf{H}^2_{\mathbb{C}}$ given for each $(\xi, \psi, u, v) \in \partial(\mathfrak{C})$ by

$$\widetilde{\Xi}(\xi, \psi, u, v) = \left(-e^{\xi + i\psi}, (u + iv)\sqrt{2\cos(\psi)}e^{(\xi + i\psi)/2}\right) \tag{6.1}$$

is a differentiable one–to–one transformation of $\partial\mathfrak{C}$ onto $\partial\mathbf{H}^2_{\mathbb{C}}$ with differentiable inverse.

The proof of this Lemma runs in the same lines as this of Lemma 4.1 and thus we leave the details for the reader.

By letting $z = ie^{-3i\eta/2}$, where now $\eta \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ we obtain the obvious

Corollary 6.2. *In the cylindrical model of the boundary of complex hyperbolic plane, the boundary of the standard pack P_κ is defined by the equation $\eta = \kappa\xi \mod (2\pi/3)$.*

We shall also need

Lemma 6.3. *In the cylindrical model of the boundary of complex hyperbolic plane, the contact form* ω *of* $\partial\mathbf{H}^2_{\mathbb{C}}$ *is given by*

$$\omega = e^{\xi}\cos(\psi)\left(\tan(\psi)d\xi + 3d\eta\right). \tag{6.2}$$

The proof of this Lemma follows immediately after we write the relation

$$\omega = -\frac{1}{2}\mathbb{J}d\rho$$

in terms of cylindrical coordinates by using Equations 4.3.

6.2. $\mathcal{C}^1$ **families of packs.** Let $\{P_\tau\}$ be a family of packs indexed by $\tau \in [0,1]$, with corresponding loxodromic elements C_τ. The family $\{P_\tau\}$ shall be called $\mathcal{C}^1$ if $C(\tau) = C_\tau$ is $\mathcal{C}^1$ in τ. This is equivalent to say that the fixed points $r(\tau)$, $a(\tau)$ and the complex lengths $\lambda(\tau) = l(\tau) + i\theta(\tau)$ of $C(\tau)$ are $\mathcal{C}^1$ functions of τ. It follows that $\kappa_\tau = \kappa(\tau) = \theta(\tau)/l(\tau)$ is also a $\mathcal{C}^1$ function.

Theorem 6.4. *Let* $\{P_\tau\}$*,* $\tau \in [0,1]$ *be a* $\mathcal{C}^1$ *family of packs. Then the following hold.*

(1) Associated to $\{P_\tau\}$ *there is a continuously time dependent Hamiltonian vector field* $\mathbf{b}_\tau$ *in* $\mathbf{H}^2_{\mathbb{C}}$ *generating a flow of symplectomorphisms* $\phi_{s,\tau}$ *such that* $\phi_{0,\tau}(P_0) = P_\tau$*.*

(2) The Hamiltonian function B_τ *of* $\mathbf{b}_\tau$ *is* $\mathcal{C}^\infty$ *and continuous in* τ*.*

(3) $\mathbf{b}_\tau$ *extends smoothly to* $\overline{\mathbf{H}^2_{\mathbb{C}}}$ *as a flow of contactomorphisms.*

(4) Let $M = \max_{\tau\in[0,1]} |\kappa'(\tau)| \geq 0$*. Then the one parameter family* $\phi_{s,\tau}$ *is quasiconformal and its dilation satisfies the following inequality.*

$$\| \mu_\tau \|^2_\infty \leq \tanh\left(\frac{3\sqrt{2}}{2}\, M\right). \tag{6.3}$$

Proof. The proof of this Theorem shall be given in steps. We may only consider the case where $\{P_\tau\}$, $\tau \in [0,1]$ is the family

$$P_\tau = \bigcup_{\xi\in\mathbb{R}} E^{\xi/2}(1 + i\kappa(\tau))(R_{\mathbb{R}}),$$

that is P_τ are all standard with curl factor $\kappa(\tau)$.

Step 1. Time dependent vector field. The symplectomorphism F_τ of complex hyperbolic plane mapping P_0 onto P_τ is given in cylindrical coordinates by

$$F_\tau(\xi,\psi,r,\eta) = \left(\xi, \arctan\left(\tan(\psi) - 3(\kappa(\tau)-\kappa_0)\right), r, \eta + (\kappa(\tau)-\kappa_0)\xi\right). \tag{6.4}$$

Let p denote an arbitrary point of $\mathbf{H}^2_{\mathbb{C}}$. We define the time dependent vector field in $\mathbf{H}^2_{\mathbb{C}}$ by

$$\mathbf{b}_\tau(F_\tau(p)) = \frac{d}{ds}\Big|_{s=\tau} F_s(p).$$

We now calculate $\mathbf{b}_\tau$ explicitely. In fact, relation

$$\mathbf{b}_\tau(p) = \frac{d}{ds}\Big|_{s=\tau} (F_s \circ F_\tau^{-1})(p)$$

yields after taking the derivative at τ of

$$\begin{aligned}\phi_{s,\tau}(\xi,\psi,r,\eta) &= (F_s\circ F_\tau^{-1})(\xi,\psi,r,\eta)\\ &= \left(\xi, \arctan\left(\tan(\psi) - 3(\kappa(s)-\kappa(\tau))\right), r, \eta + (\kappa(s)-\kappa(\tau))\xi\right)\end{aligned}$$

that

$$\mathbf{b}_\tau = \frac{dF_{s,\tau}}{ds}\ |_{s=\tau} = \kappa'(\tau)\left(-3\cos^2(\psi)\frac{\partial}{\partial\psi} + \xi\frac{\partial}{\partial\eta}\right). \tag{6.5}$$

Hence $\mathbf{b}_\tau$ is $\mathcal{C}^\infty$ everywhere in $\mathbf{H}^2_{\mathbb{C}}$ and by hypothesis is also continuous in τ.

Step 2. The Hamiltonian.

Lemma 6.5. *The Hamiltonian function of* $\mathbf{b}_\tau$ *is given globally by*

$$B_\tau = H_{\mathbf{b}_\tau} = -3\kappa'(\tau)\frac{\xi}{1-r^2}.$$

Proof. We calculate the inner product $i(\mathbf{b}_\tau)\Omega$. That is

$$\begin{aligned} i(\mathbf{b}_\tau)\Omega &= \Omega(\mathbf{b}_\tau,\cdot) \\ &= -3\kappa'(\tau)\left(\frac{d\xi}{1-r^2} + \frac{2\xi r dr}{(1-r^2)^2}\right) \\ &= -3\kappa'(\tau) d\left(\frac{\xi}{1-r^2}\right) \end{aligned}$$

and this proves our statement. □

Step 3. Extension to the boundary. Following Proposition 1 of [10], $\mathbf{b}_\tau$ is extended to the boundary and its flow is a flow of contactomorphisms. Without making use of that, one may observe that the extension of $\phi_{s,\tau}$ to the boundary-denoted again by the same letter-is given there by

$$\phi_{s,\tau}(\xi,\psi,\eta) = \left(\xi, \arctan\left(\tan(\psi) - 3(\kappa(s)-\kappa(\tau)\right), \eta + (\kappa(s)-\kappa(\tau))\xi)\right) \tag{6.6}$$

and is a flow of contactomorphisms. Note further that in the boundary, $\mathbf{b}_\tau$ is given in exactly the same formula as in Equation 6.5.

By Proposition 24 of [12], which also applies for time dependent vecto fields, the time dependent vector field $\mathbf{b}_\tau$ has to be of the form

$$\mathbf{b}_\tau = \frac{i}{2}\left((\overline{Z}b_\tau)Z - (Zb_\tau)\overline{Z}\right) + b_\tau T$$

where b_τ is a real valued $\mathcal{C}^\infty$ function. Again, we don't have to use this Proposition; the interested reader may verify this formula for $\mathbf{b}_\tau$ by direct calculations.

Corollary 6.6. *In Heisenberg coordinates,*

$$b_\tau = \frac{3}{2}\kappa'(\tau)|\zeta|^2\log\left(|\zeta|^4+t^2\right).$$

Proof. We have $\omega(\mathbf{b}_\tau) = b_\tau\omega(T)$ and since additionally $\omega(T)=1$ we obtain

$$\begin{aligned} b_\tau = \omega(\mathbf{b}_\tau) &= 3\kappa'(\tau)\ \xi e^\xi\cos(\psi) \\ &= -3\kappa'(\tau)\ x_1\log(|z_1|) \\ &= \frac{3}{2}\kappa'(\tau)\ |\zeta|^2\log\left(|\zeta|^4+t^2\right). \end{aligned}$$

□

Step 4. Complex dilation. Calculation of the complex dilation is based on the following.

Lemma 6.7. *If b_τ is as in Corollary 6.6 then*

$$|ZZb_\tau| \le 3|\kappa'(\tau)|.$$

Proof. We set

$$\widetilde{b} = |\zeta|^2 \log\left(|\zeta|^4 + t^2\right) = f_1 \log(f_2)$$

and let also

$$Z = \frac{\partial}{\partial \zeta} + i\overline{\zeta}\frac{\partial}{\partial t}.$$

Then we only have to prove that $|ZZ\widetilde{b}| \le 2$. We have

$$\begin{aligned} Z\widetilde{b} &= Zf_1\ \log(f_2) + \frac{f_1}{f_2}\ Zf_2, \\ ZZ\widetilde{b} &= ZZf_1\ \log(f_2) + \frac{2}{f_2}\ (Zf_1)(Zf_2) \\ &\quad -\frac{f_1}{f_2^2}\ (Zf_2)^2 + \frac{f_1}{f_2}\ ZZf_2. \end{aligned}$$

Now,

$$\begin{aligned} Zf_1 &= \overline{\zeta}, \\ Zf_2 &= 2\overline{\zeta}(|\zeta|^2 + it), \\ ZZf_1 &= 0, \\ ZZf_2 &= 0. \end{aligned}$$

Thus

$$\begin{aligned} ZZ\widetilde{b} &= \frac{2}{f_2}\ Zf_1 Zf_2 - \frac{f_1}{f_2^2}\ (Zf_2)^2 \\ &= \frac{4\overline{\zeta}^2}{|\zeta|^2 - it}\left(1 - \frac{|\zeta|^2}{|\zeta|^2 - it}\right) \\ &= -\frac{4\overline{\zeta}^2 t}{(|\zeta|^2 - it)^2} \end{aligned}$$

and therefore,

$$|ZZ\widetilde{b}| = \frac{4|\zeta|^2|t|}{||\zeta|^2 - it|^2} = \frac{4|\zeta|^2|t|}{|\zeta|^4 + t^2} \le 2.$$

□

Hence, from (3) of Proposition 2.4 we obtain that F_τ is $K-$quasiconformal with

$$K + K^{-1} \le 2e^{3\sqrt{2}M\tau} \le 2e^{3\sqrt{2}M}.$$

Finally, if $\| \mu_\tau \|$ is the complex dilation of F_τ then

$$\| \mu_\tau \| \le \frac{K-1}{K+1}$$

and therefore

$$\| \mu_\tau \|_\infty \le \frac{e^{3\sqrt{2}M} + \sqrt{e^{6\sqrt{2}M} - 1} - 1}{e^{3\sqrt{2}M} + \sqrt{e^{6\sqrt{2}M} - 1} + 1},$$

which is equivalent to Equation 6.3. □

7. Differential Geometry

In this section we simultaneously study the differential geometry of bisectors and packs, in the same spirit of the discussion carried out in section 5.5 of [6]. For the notation we use, we refer the author to the general textbooks [3] and [9].

We start by expressing the Bergman-Kähler Riemaniann tensor of complex hyperbolic plane in terms of cylindrical coordinates (ξ, ψ, r, η) of complex hyperbolic plane.

Proposition 7.1. *In the cylindrical model of complex hyperbolic plane, the Bergman-Kähler metric tensor* $\mathbf{g}$ *is given by*

$$\begin{aligned}\mathbf{g} &= \frac{1}{(1-r^2)^2}\Big((1+\tan^2(\psi)-r^2)d\xi^2 + 6r^2\tan(\psi)d\xi d\eta + 9r^2 d\eta^2 \\ &\quad +(1-r^2)(1+\tan^2(\psi))d\psi^2 + 4dr^2\Big).\end{aligned} \tag{7.1}$$

Proof. Denote by J the matrix of Equation 4.3 and denote also by Ω, G the matrices of the symplectic form and the Riemannian product respectively, where Ω is given by Proposition 5.2. Since $G = \Omega \cdot J$, we obtain

$$G = \frac{1}{(1-r^2)^2}\begin{pmatrix} 1+\tan^2(\psi)-r^2 & 0 & 0 & 3\,r^2\tan(\psi) \\ 0 & (1-r^2)\,(1+\tan^2(\psi)) & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 3r^2\tan(\psi) & 0 & 0 & 9\,r^2 \end{pmatrix} \tag{7.2}$$

and formula 7.1 follows. □

The inverse matrix $G^{-1} = (g^{ij})$ is given by

$$G^{-1} = \frac{(1-r^2)}{1+\tan^2(\psi)}\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3}\tan(\psi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (1-r^2)(1+\tan^2(\psi)) & 0 \\ -\frac{1}{3}\tan(\psi) & 0 & 0 & \frac{1+\tan^2(\psi)-r^2}{9r^2} \end{pmatrix}. \tag{7.3}$$

7.1. **Orthogonality.** We shall prove

Theorem 7.2. *Let B be a κ_1-bisector with complex spine $L_{\mathbb{C}}$ and suppose that P is a pack with curl factor κ_2 whose complex spine is also $L_{\mathbb{C}}$. Then there exist codimension one foliations $\mathcal{F}_B$ and $\mathcal{F}_P$ of $\mathbf{H}^2_{\mathbb{C}}$ such that*

(1) The leaves of $\mathcal{F}_B$ (resp. of $\mathcal{F}_P$) are diffeomorphic to B (resp. to P).
(2) $\mathcal{F}_B$ is orthogonal to $\mathcal{F}_P$.

The proof of this Theorem is an immediate consequence of the following Proposition.

Proposition 7.3. *Let $(\kappa_1, \kappa_2) \in \mathbb{R}^2$ and let also B be a κ_1-bisector with complex spine $L_{\mathbb{C}}$ and P be a pack with curl factor κ_2 whose complex spine is also $L_{\mathbb{C}}$. Let N_B and N_P be vector fields normal to B and P respectively. Then $N_B \perp N_P$. In particular, if $\mathfrak{b}_{\kappa_1}$ and $\mathfrak{p}_{\kappa_2}$ are defining functions for B and P respectively, then* $\mathrm{grad}\mathfrak{b}_{\kappa_1} \perp \mathrm{grad}\mathfrak{p}_{\kappa_2}$.

Proof. We may restrict ourselves to the case where $B = B_\kappa$ with defining function $\mathfrak{b}_{\kappa_1} = \psi + \arctan(3\kappa_1)$ and $P = P_{\kappa_2}$ with defining function $\mathfrak{p}_{\kappa_2} = \eta - \kappa_2\xi$. Then by using Equation 7.3 we have

$$\begin{aligned}\mathbf{g}(\mathrm{grad}\mathfrak{b}_{\kappa_1}, \mathrm{grad}\mathfrak{p}_{\kappa_2}) &= \mathbf{g}(d\mathfrak{b}_{\kappa_1}, d\mathfrak{p}_{\kappa_2})\\ &= \mathbf{g}(d\psi, d\eta - \kappa_2 d\xi)\\ &= \mathbf{g}(d\psi, d\eta) - \kappa_2\mathbf{g}(d\psi, d\xi)\\ &= 0 - \kappa_2 \cdot 0\\ &= 0.\end{aligned}$$

□

As a Corollary, we also have

Corollary 7.4. *Let X_B and X_P be the Hamiltonian vector fields to B and P respectively, where B and P are as in Proposition 7.3. Then $X_B \perp X_P$.*

Proof. By restricting ourselves again to the case where $B = B_{\kappa_1}$ and $P = P_{\kappa_2}$ we have

$$\mathbf{g}(X_B, X_P) = \mathbf{g}(\mathbb{J}\mathrm{grad}\mathfrak{b}_{\kappa_1}, \mathbb{J}\mathrm{grad}\mathfrak{p}_{\kappa_2}) = \mathbf{g}(\mathrm{grad}\mathfrak{b}_{\kappa_1}, \mathrm{grad}\mathfrak{p}_{\kappa_2}) = 0.$$

□

7.2. **Comparative geometry of packs and bisectors.** In this section we shall prove

Theorem 7.5. *Let P be a pack with curl factor κ and B be a $\kappa-$bisector. Then*

1. *P is a minimal submanifold of $\mathbf{H}^2_{\mathbb{C}}$ with zero Gauss-Kronecker curvature.*
2. *B is a submanifold of $\mathbf{H}^2_{\mathbb{C}}$ with zero Gauss-Kronecker curvature. Moreover, it is minimal if and only if $\kappa = 0$.*

Our strategy for the proof is the following. Instead of studying the geometrical aspects of packs and bisectors induced from the Bergman metric $\mathbf{g}$ of $\mathbf{H}^2_{\mathbb{C}}$, we study the pulled-back metric $\mathbf{g}_\kappa$ of $\mathbf{H}^2_{\mathbb{C}}$ which arises from the action of F_κ. We next proceed by constructing and orthonormal frame of vector fields X_i, $i = 1, \dots 4$ for this metric, which is such that

$$X_2 = \frac{\mathrm{grad}\mathfrak{b}_0}{\| \mathrm{grad}\mathfrak{b}_0 \|_\kappa}, \quad X_4 = \frac{\mathrm{grad}\mathfrak{p}_0}{\| \mathrm{grad}\mathfrak{p}_0 \|_\kappa}$$

that is, X_2 is a unit normal vector field to the standard bisector B_0 and X_4 is a unit normal vector field to he standard flat pack P_0. Following standard differential geometric procedures, we calculate next the second fundamental forms and the sectional curvatures of B_0 and P_0 in the $\mathbf{g}_\kappa$ metric. This will prove our Theorem.

Proof. The proof will be given in steps.

Step 1. The pulled–back metric. Let

$$F_\kappa(\xi, \psi, r, \eta) = (\xi, \arctan(\tan(\psi) - 3\kappa), r, \eta + \kappa\xi),$$

be the symplectomorphism as in Equation 5.3. By calculating straightforwrdly we obtain

$$\begin{aligned}\mathbf{g}_\kappa = F_\kappa^*\mathbf{g} &= \frac{1}{(1-r^2)^2}\Big((\tan^2(\psi) + (1-r^2)(1 + 9\kappa^2 - 6\kappa\tan(\psi)))d\xi^2\\ &\quad +6r^2\tan(\psi)d\xi d\eta + 9r^2 d\eta^2 + (1-r^2)\frac{(1+\tan^2(\psi))^2}{1+(\tan(\psi)-3\kappa)^2}d\psi^2 + 4dr^2\Big).\end{aligned}$$

In order to clearly show the consistency of our results with those of Goldman, we find suitable to set

$$\sigma = \tan(\psi), \quad \tau = 2\tanh^{-1}(r).$$

In this way, we may write $\mathbf{g}_\kappa$ in terms of ξ, σ, τ, η as follows.

$$\begin{aligned}\mathbf{g}_\kappa \;=\;& \cosh^2(\tau/2)\left((\sigma-3\kappa)^2+\sigma^2\sinh^2(\tau/2)+1\right)d\xi^2\\ &+6\sigma\cosh^2(\tau/2)\sinh^2(\tau/2)d\xi d\eta+9\cosh^2(\tau/2)\sinh^2(\tau/2)d\eta^2\\ &+\frac{\cosh^2(\tau/2)}{1+(\sigma-3\kappa)^2}d\sigma^2+d\tau^2.\end{aligned} \tag{7.4}$$

Let

$$f=\frac{\operatorname{sech}^2(\tau/2)}{1+(\sigma-3\kappa)^2},\quad g=\sinh^2(\tau/2)\cosh^2(\tau/2),\quad h=1+\sigma^2 fg.$$

Then,

$$G_\kappa=\begin{pmatrix} f^{-1}h & 0 & 0 & 3\sigma g\\ 0 & f\cosh^4(\tau/2) & 0 & 0\\ 0 & 0 & 1 & 0\\ 3\sigma g & 0 & 0 & 9g\end{pmatrix}.$$

Thus $\det(G_\kappa)=9\cosh^6(\tau/2)\sinh^2(\tau/2)$ and

$$G_\kappa^{-1}=\begin{pmatrix} f & 0 & 0 & -\frac{1}{3}\sigma f\\ 0 & f^{-1}\operatorname{sech}^4(\tau/2) & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{1}{3}\sigma f & 0 & 0 & \frac{1}{9}g^{-1}h\end{pmatrix}.$$

A frame of orthonormal vector fields for the metric $\mathbf{g}_\kappa$ is given by

$$\begin{aligned} X_1 &= f^{1/2}h^{-1/2}\frac{\partial}{\partial\xi},\\ X_2 &= f^{-1/2}\operatorname{sech}^2(\tau/2)\frac{\partial}{\partial\sigma},\\ X_3 &= \frac{\partial}{\partial\tau},\\ X_4 &= -\sigma f g^{1/2}h^{-1/2}\frac{\partial}{\partial\xi}+\frac{1}{3}g^{-1/2}h^{1/2}\frac{\partial}{\partial\eta}.\end{aligned}$$

The corresponding orthonormal coframe is

$$\begin{aligned} \phi^1 &= f^{-1/2}h^{1/2}d\xi+3\sigma g f^{1/2}h^{-1/2}d\eta,\\ \phi^2 &= f^{1/2}\cosh^2(\tau/2)d\sigma,\\ \phi^3 &= d\tau,\\ \phi^4 &= 3g^{1/2}h^{-1/2}d\eta.\end{aligned}$$

It is clear that the vector fields X_2, X_4 are the unit normal vector fields to the standard bisector B_0 with defining equation $\mathfrak{b}_0=\sigma=0$ and to the standard flat pack P_0 with defining function $\mathfrak{p}_0=\eta=0$ respectively.

Step 2. Second fundamental forms. We calculate here the second fundamental forms $\mathbf{II}_\kappa(B_0)$ of B_0 and $\mathbf{II}_\kappa(P_0)$ of P_0. For this, let $\mathbf{II}_\kappa^\sigma$ and $\mathbf{II}_\kappa^\eta$ be the second fundamental forms of the level sets of the coordinate functions σ and η respectively. Then,

$$\begin{aligned}\mathbf{II}_\kappa^\sigma(X_j,X_k) &= \mathbf{g}_\kappa(X_2,\nabla_{X_j}X_k),\ j,k=1,3,4,\\ \mathbf{II}_\kappa^\eta(X_j,X_k) &= \mathbf{g}_\kappa(X_4,\nabla_{X_j}X_k),\ j,k=1,2,3,\end{aligned}$$

where of course

$$\mathbf{g}_\kappa(X_i, \nabla_{X_j} X_k) = -\frac{1}{2}\left\{\mathbf{g}_\kappa([X_k, X_i], X_j) + \mathbf{g}_\kappa([X_j, X_i], X_k) + \mathbf{g}_\kappa([X_k, X_j], X_i)\right\}, \; i = 2, 4.$$

Simple calculations show that

$$\mathbf{II}^\sigma_\kappa = \begin{pmatrix} \widehat{a}_{11} & 0 & \widehat{a}_{14} \\ 0 & 0 & 0 \\ \widehat{a}_{14} & 0 & 0 \end{pmatrix},$$

where

$$\widehat{a}_{11} = \frac{1}{2} f^{-1/2} \mathrm{sech}^2(\tau/2) \frac{\partial \log(f/h)}{\partial \sigma},$$
$$\widehat{a}_{14} = -\frac{1}{2} g^{1/2} \mathrm{sech}^2(\tau/2) \left(1 + \sigma \frac{\partial \log(f/h)}{\partial \sigma}\right).$$

Where,

$$\frac{\partial \log(f/h)}{\partial \sigma} = \frac{6\kappa \cosh^2(\tau/2) - 2\sigma \cosh(\tau)}{1 + (\sigma - 3\kappa)^2 + \sigma^2 \sinh^2(\tau/2)}. \tag{7.5}$$

On B_0 we have $\sigma = 0$. Thus, if $\mathbf{II}(B_\kappa)$ is the second fundamental form of B_κ with respect to the metric $\mathbf{g}$, it follows

$$\mathbf{II}_\kappa(B_0) = \mathbf{II}(B_\kappa) = \begin{pmatrix} 3\kappa(1 + 9\kappa^2)^{-1/2} \cosh(\tau/2) & 0 & -\frac{1}{2}\tanh(\tau/2) \\ 0 & 0 & 0 \\ -\frac{1}{2}\tanh(\tau/2) & 0 & 0 \end{pmatrix}.$$

Therefore, B_κ is minimal if and only if $\kappa = 0$.

On the other hand we have

$$\mathbf{II}^\eta_\kappa = \begin{pmatrix} 0 & \widetilde{a}_{12} & \widetilde{a}_{13} \\ \widetilde{a}_{12} & 0 & 0 \\ \widetilde{a}_{13} & 0 & 0 \end{pmatrix}$$

where

$$\widetilde{a}_{12} = \frac{1}{2} \tanh(\tau/2) \left(1 + \sigma \frac{\partial \log(f/h)}{\partial \sigma}\right),$$
$$\widetilde{a}_{13} = \frac{1}{2} \sigma (fg)^{1/2} \frac{\partial \log(fg/h)}{\partial \tau}.$$

Here,

$$\frac{\partial \log(fg/h)}{\partial \tau} = \cosh(\tau/2)\, \frac{1 + (\sigma - 3\kappa)^2}{1 + (\sigma - 3\kappa)^2 + \sigma^2 \sinh^2(\tau/2)}.$$

On P_0 we have $\eta = 0$ and thus if $\mathbf{II}(P_\kappa)$ is the second fundamental form of P_κ with repect to the metric $\mathbf{g}$ we obtain

$$\mathbf{II}(P_\kappa) = \mathbf{II}_\kappa(P_0) = \mathbf{II}^\eta_\kappa$$

since $\mathbf{II}^\eta_\kappa$ depends only on σ, τ. This also proves minimality of P_κ. Finally, since both determinants of $\mathbf{II}(B_\kappa)$ and $\mathbf{II}(P_\kappa)$ vanish, we have that both B_κ and P_κ are submanifolds wof zero Gauss-Kronecker curvature.

□

7.2.1. *Curvature.* Our goal in this section is to study the curvature of the submanifolds B_κ and P_κ with respect to the metric $\mathbf{g}$. For this, it suffices to study the curvatures of the submanifolds B_0 and P_0 with respect to the metric $\mathbf{g}_\kappa$. Thus we will resrict ourselves each time to the cases where $\sigma = 0$ and $\eta = 0$ respectively.

We begin with the case of B_0. This is given by $\sigma = 0$ and therefore from the coframe constructed in the first step of the proof of Theorem 7.5 we obtain the corresponding coframe

$$\begin{aligned}
\widehat{\phi}^1 &= \cosh(\tau/2)(1+9\kappa^2)^{-1/2}d\xi,\\
\widehat{\phi}^3 &= d\tau,\\
\widehat{\phi}^4 &= 3\cosh(\tau/2)\sinh(\tau/2)d\eta,
\end{aligned}$$

which is associated to the restriction of the metric $\mathbf{g}_\kappa$ on the submanifold B_0. Now,

$$\begin{aligned}
d\widehat{\phi}^1 &= \frac{1}{2}\tanh(\tau/2)\ \widehat{\phi}^3 \wedge \widehat{\phi}^1,\\
d\widehat{\phi}^3 &= 0,\\
d\widehat{\phi}^4 &= \coth(\tau)\ \widehat{\phi}^3 \wedge \widehat{\phi}^4.
\end{aligned}$$

If $\widehat{\Theta} = [\widehat{\theta}_i^j],\ i,j = 1,3,4$, is the matrix of the conection form, then from structural equations

$$d\widehat{\phi}^i = -\sum_i \widehat{\theta}_i^j \wedge \widehat{\phi}^j$$

we obtain

$$\widehat{\Theta} = \begin{pmatrix} 0 & \frac{1}{2}\tanh(\tau/2)\ \widehat{\phi}^1 & 0\\ -\frac{1}{2}\tanh(\tau/2)\ \widehat{\phi}^1 & 0 & -\coth(\tau)\ \widehat{\phi}^4\\ 0 & \coth(\tau)\ \widehat{\phi}^4 & 0 \end{pmatrix}.$$

Let $\widehat{\Omega}$ be the matrix of the curvature form. Then from structural equations

$$\widehat{\Omega} = d\widehat{\Theta} + \widehat{\Theta} \wedge \widehat{\Theta}$$

we have

$$\widehat{\Omega} = \begin{pmatrix} 0 & -\frac{1}{4}\ \widehat{\phi}^1 \wedge \widehat{\phi}^3 & -\frac{1}{4}(1+\tanh^2(\tau/2))\ \widehat{\phi}^1 \wedge \widehat{\phi}^4\\ -\frac{1}{4}\ \widehat{\phi}^3 \wedge \widehat{\phi}^1 & 0 & -\widehat{\phi}^3 \wedge \widehat{\phi}^4\\ -\frac{1}{4}(1+\tanh^2(\tau/2))\ \widehat{\phi}^4 \wedge \widehat{\phi}^4 & -\widehat{\phi}^4 \wedge \widehat{\phi}^3 & 0 \end{pmatrix}.$$

The sectional curvatures of the coordinate 2–planes are therefore

$$\widehat{K}_{13} = -\frac{1}{4}, \quad \widehat{K}_{14} = -\frac{1}{4}\left(1+\tanh^2(\tau/2)\right), \quad \widehat{K}_{34} = -1.$$

Observe here that $\widehat{K}_{ij}$ do not depend on κ. (Compare also with [6], p.190.)

We next proceed to the case of P_0. Here is $\eta = 0$ and by setting

$$\begin{aligned}
k &= (f^{-1}+\sigma^2 g)^{-1/2}(f^{-1}+\sigma^2 g+3\kappa\sigma)\\
&= (fh)^{-1/2}(h+3\kappa\sigma f)\\
&= (h/f)^{-1/2}(h/f+3\kappa\sigma).
\end{aligned}$$

we obtain the corresponding coframe

$$\begin{aligned}\widetilde{\phi}^1 &= k\, d\xi,\\ \widetilde{\phi}^2 &= f^{1/2}\cosh^2(\tau/2)\, d\sigma,\\ \widetilde{\phi}^3 &= d\tau,\end{aligned}$$

for the submanifold P_0 of $(\mathbf{H}^2_{\mathbb{C}}, \mathbf{g}_\kappa)$. Now,

$$\begin{aligned} d\widetilde{\phi}^1 &= \frac{\partial \log(k)}{\partial\sigma} f^{-1/2}\mathrm{sech}^2(\tau/2)\, \widetilde{\phi}^2\wedge\widetilde{\phi}^1 + \frac{\partial\log(k)}{\partial\tau}\,\widetilde{\phi}^3\wedge\widetilde{\phi}^1,\\ d\widetilde{\phi}^2 &= -\frac{1}{2}\tanh(\tau/2)\,\widetilde{\phi}^2\wedge\widetilde{\phi}^3,\\ d\widehat{\phi}^3 &= 0.\end{aligned}$$

Where,

$$\begin{aligned}\frac{\partial\log(k)}{\partial\tau} &= -\frac{1}{2}\frac{h-3\kappa\sigma f}{h+3\kappa\sigma f}\,\frac{\partial(\log(f/h))}{\partial\tau},\\ \frac{\partial\log(k)}{\partial\sigma} &= -\frac{1}{2}\frac{h-3\kappa\sigma f}{h+3\kappa\sigma f}\,\frac{\partial(\log(f/h))}{\partial\sigma} + \frac{3\kappa f}{h+3\kappa\sigma f}\end{aligned}$$

with $\frac{\partial(\log(f/h))}{\partial\sigma}$ given by Equation 7.5 and

$$\frac{\partial(\log(f/h))}{\partial\tau} = -\tanh(\tau/2)\frac{1+(\sigma-3\kappa)^2+\sigma^2\cosh(\tau)}{1+(\sigma-3\kappa)^2+\sigma^2\sinh^2(\tau/2)}. \tag{7.6}$$

Denote by $\widetilde{\Theta} = [\widetilde{\theta}^j_i],\ i,j=1,2,3,$ the matrix of the connection form. From structural equations

$$d\widetilde{\phi}^i = -\sum_i \widehat{\theta}^j_i\wedge\widehat{\phi}^j$$

we obtain

$$\widetilde{\Theta} = \begin{pmatrix} 0 & \frac{\partial\log(k)}{\partial\sigma} f^{-1/2}\mathrm{sech}^2(\tau/2)\,\widetilde{\phi}^1 & \frac{\partial\log(k)}{\partial\tau}\,\widetilde{\phi}^1\\ -\frac{\partial\log(k)}{\partial\sigma} f^{-1/2}\mathrm{sech}^2(\tau/2)\,\widetilde{\phi}^1 & 0 & \frac{1}{2}\tanh(\tau/2)\,\widetilde{\phi}^2\\ -\frac{\partial\log(k)}{\partial\tau}\,\widetilde{\phi}^1 & -\frac{1}{2}\tanh(\tau/2)\,\widetilde{\phi}^2 & 0\end{pmatrix}.$$

The matrix $\widetilde{\Omega}$ of the curvature form given from the structural equations

$$\widetilde{\Omega} = d\widetilde{\Theta} + \widetilde{\Theta}\wedge\widetilde{\Theta}$$

is quite complicated and we shall not write it down explicitely. However, the sectional curvatures of the coordinate 2–planes may be expressed as follows.

$$\begin{aligned}\widetilde{K}_{12} &= -\mathrm{sech}^2(\tau/2)(1+(\sigma-3\kappa)^2)\left(\frac{\partial^2\log(k)}{\partial\sigma^2} + \left(\frac{\partial\log(k)}{\partial\sigma}\right)^2\right)\\ &\quad -\mathrm{sech}^2(\tau/2)(\sigma-3\kappa)\frac{\partial\log(k)}{\partial\sigma},\\ \widetilde{K}_{13} &= -\left(\frac{\partial^2\log(k)}{\partial\tau^2} + \left(\frac{\partial\log(k)}{\partial\tau}\right)^2\right),\\ \widetilde{K}_{23} &= -\frac{1}{4}.\end{aligned}$$

In the case of the flat pack P_0 we have

$$\begin{aligned}
\widetilde{K}_{12} &= -\frac{1}{4}\,\frac{(1-\tanh^4(\tau/2))\left((1-\tanh^2(\tau/2))(1+\sigma^2)+2\sigma^4\right)}{(1+\sigma^2-\tanh^2(\tau/2))^2},\\
\widetilde{K}_{13} &= -\frac{1}{4}\,\frac{(3-5\sigma^2)\tanh^4(\tau/2)+2(2\sigma^4+\sigma^2-2)\tanh^2(\tau/2)+1}{(1+\sigma^2-\tanh^2(\tau/2))^2},\\
\widetilde{K}_{23} &= -\frac{1}{4}.
\end{aligned}$$

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