M1226
Πιθανότητες ΙΙ
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Λύσεις 2

11/10/12

1) $F(x) = P(X \le x) = \sum_{k=1}^{[x]} P(X=k) = \sum_{k=1}^{[x]} p(1-p)^{k-1}$

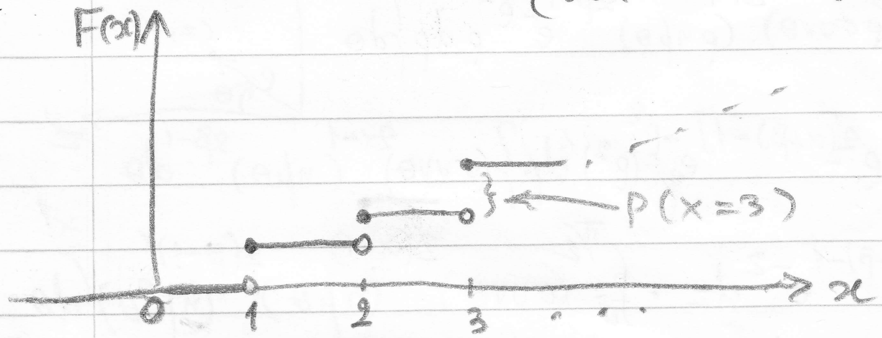
$= \sum_{k=1}^{\infty} p(1-p)^{k-1} - \sum_{k=[x]+1}^{\infty} p(1-p)^{k-1} =$

$= 1 - p \sum_{m=[x]}^{\infty} (1-p)^{m} = 1 - p(1-p)^{[x]} \sum_{n=0}^{\infty} (1-p)^{n}$

$m \equiv k-1$ $\qquad$ $n \equiv m-[x]$

$= 1 - p(1-p)^{[x]} \frac{1}{1-(1-p)} = 1 - (1-p)^{[x]}, \; x \in \mathbb{R}_+$

(και $= 0$ για $x < 0$.)



2)(α) $\Gamma(\alpha+1) = \int_0^{\infty} x^{(\alpha+1)-1} e^{-x}\,dx =$

$= \int_0^{\infty} x^{\alpha} e^{-x}\,dx = \int_0^{\infty} x^{\alpha} (-e^{-x})'\,dx =$

$= -x^{\alpha} e^{-x}\Big|_0^{\infty} - \int_0^{\infty} (-e^{-x})(\alpha x^{\alpha-1})\,dx$

$(\alpha>0)$

$= 0 + \alpha \int_0^{\infty} x^{\alpha-1} e^{-x}\,dx = \alpha\,\Gamma(\alpha)$

$\forall \alpha > 0$

$\Rightarrow \Gamma(n) = n\,\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) =$

$= n(n-1)(n-2)\cdots 2\,\underbrace{\Gamma(1)}_{=1} = (n-1)!, \; \forall n \in \mathbb{N}$

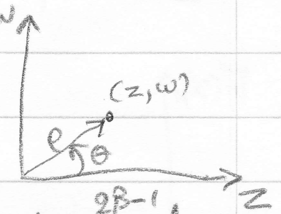
(ελέγξτε το)

(β) $\Gamma(1/2) = \int_0^\infty x^{\frac{1}{2}-1} e^{-x}\,dx = \int_0^\infty x^{-1/2} e^{-x}\,dx =$
$= \sqrt{\pi}$ από Άσκηση 1.4.

Οπότε, $\Gamma(\frac{5}{2}) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\Gamma(\frac{1}{2}+1) =$
$= \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(1/2) = 3\sqrt{\pi}/4$.

3) Κατ' αρχάς, θέτοντας $x = z^2$, έχουμε:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx = \int_0^\infty (z^2)^{\alpha-1} e^{-z^2} 2z\,dz = $$
$$= 2\int_0^\infty z^{2\alpha-1} e^{-z^2}\,dz$$
$$\Rightarrow \Gamma(\alpha)\Gamma(\beta) = 4\int_0^\infty\int_0^\infty z^{2\alpha-1} w^{2\alpha-1} e^{-(z^2+w^2)}\,dz\,dw$$
$$= 4\int_0^{\pi/2}\left\{\int_0^\infty (\rho\,\sigma\upsilon\nu\theta)^{2\alpha-1} (\rho\,\eta\mu\theta)^{2\beta-1} e^{-\rho^2} \rho\,d\rho\right\} d\theta$$



$$= 2\int_0^{\pi/2}\left\{\int_0^\infty \rho^{2[(\alpha+\beta)-1]} e^{-\rho^2} (\rho^2)'\,d\rho\right\}(\sigma\upsilon\nu\theta)^{2\alpha-1}(\eta\mu\theta)^{2\beta-1}\,d\theta$$
$$\overset{\rho^2=z}{=} \int_0^\infty z^{(\alpha+\beta)-1} e^{-z}\,dz \cdot \int_0^{\pi/2} (\sigma\upsilon\nu\theta)^{2(\alpha-1)} (\eta\mu\theta)^{2(\beta-1)} (\eta\mu^2\theta)'\,d\theta$$
$$\overset{y=\eta\mu^2\theta}{=} \Gamma(\alpha+\beta)\int_0^1 (1-y)^{\alpha-1} y^{\beta-1}\,dy \overset{t=1-y}{=} \Gamma(\alpha+\beta)\int_1^0 t^{\alpha-1}(1-t)^{\beta-1}(-1)\,dt$$
$$= \Gamma(\alpha+\beta)\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \Gamma(\alpha+\beta)\,B(\alpha,\beta).$$
$$\Rightarrow B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \forall\, \alpha,\beta > 0.$$

4) $X \sim \mathcal{NB}:(\kappa,p)$: $M_X(t) = E e^{-tX} = \sum_{j=\kappa}^\infty e^{-tj}\binom{j-1}{\kappa-1}p^\kappa(1-p)^{j-\kappa}$

$$= p^\kappa e^{-\kappa t}\sum_{j=\kappa}^\infty \binom{j-1}{\kappa-1}\left[(1-p)e^{-t}\right]^{j-\kappa} \underset{\substack{\text{Άσκηση 1.1} \\ \text{με } q=(1-p)e^{-t} \\ <1}}{\overset{\Sigma}{=}}$$
$$= p^\kappa e^{-\kappa t}\left(1-(1-p)e^{-t}\right)^{-\kappa} \quad \forall\, t > \log(1-p)$$