M1226
Πιθανότητες II
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Λύσεις 7

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1) Κατ' αρχάς $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy =$

$$= \int_0^{1-x} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1}\,dy\ \mathbb{1}(x \in (0,1))$$

$$= C\, x^{\alpha-1} (1-x)^{\gamma-1+1+\beta-1} \int_0^1 z^{\beta-1} (1-z)^{\gamma-1}\,dz \quad \left(\text{όπου } z = \frac{y}{1-x}\right)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}\cancel{\Gamma(\gamma)}} x^{\alpha-1}(1-x)^{\beta+\gamma-1} \cdot \frac{\cancel{\Gamma(\beta)}\cancel{\Gamma(\gamma)}}{\Gamma(\beta+\gamma)} =$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} x^{\alpha-1}(1-x)^{\beta+\gamma-1} \cdot \mathbb{1}(0<x<1)$$

δηλ., $X \sim \mathrm{Beta}(\alpha, \beta+\gamma)$
και ομοίως: $Y \sim \mathrm{Beta}(\beta, \alpha+\gamma)$ $\Rightarrow$

$$\Rightarrow EX = \frac{\alpha}{\alpha+\beta+\gamma},\quad EY = \frac{\beta}{\alpha+\beta+\gamma},\quad \mathrm{Var}X = \frac{\alpha(\beta+\gamma)}{(\alpha+\beta+\gamma)^2(\alpha+\beta+\gamma+1)},\quad \mathrm{Var}Y = \frac{\beta(\alpha+\gamma)}{(\alpha+\beta+\gamma)^2(\alpha+\beta+\gamma+1)}.$$

Ενώ: $EXY = \iint_{\mathbb{R}\,\mathbb{R}} xy\, f_{X,Y}(x,y)\,dx\,dy = C \int_0^1\!\!\int_0^1 x^{\alpha} y^{\beta} (1-x-y)^{\gamma-1}\, \mathbb{1}(x+y<1)\,dx\,dy$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\cancel{\Gamma(\gamma)}} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\cancel{\Gamma(\gamma)}}{\Gamma(\alpha+\beta+\gamma+2)} = \frac{\alpha\cancel{\Gamma(\alpha)}\,\beta\cancel{\Gamma(\beta)}\,\cancel{\Gamma(\alpha+\beta+\gamma)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}(\alpha+\beta+\gamma+1)(\alpha+\beta+\gamma)\cancel{\Gamma(\alpha+\beta+\gamma)}}$$

$$\Rightarrow EXY = \frac{\alpha\beta}{(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)} \Rightarrow \mathrm{cov}(X,Y) = \frac{-\alpha\beta}{(\alpha+\beta+\gamma)^2(\alpha+\beta+\gamma+1)}$$

$$\Rightarrow \rho(X,Y) = -\sqrt{\frac{\alpha\beta}{(\beta+\gamma)(\alpha+\gamma)}}$$

2) Παρατηρείστε ότι: $f_{X,Y}(x,y) = \frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)\phi(x) = \frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\phi(y)$

$\Rightarrow f_X(x) = \int f_{X,Y}(x,y)\,dy = \phi(x)$
και $f_Y(y) = \int f_{X,Y}(x,y)\,dx = \phi(y)$ $\Rightarrow$

$\Rightarrow X, Y \sim N(0,1)$ αλλά δεν είναι ανεξ.

$\Rightarrow EX = EY = 0,\ \mathrm{Var}X = \mathrm{Var}Y = 1.$

Και: $\mathrm{cov}(X,Y) = EXY = \iint xy\, f_{X,Y}(x,y)\,dx\,dy =$

$$= \int_{\mathbb{R}} \Big\{ \underbrace{\int_{\mathbb{R}} \frac{(y-\rho x)}{\sqrt{1-\rho^2}} \phi\Big(\frac{y-\rho x}{\sqrt{1-\rho^2}}\Big) dy}_{=0} \Big\} x \phi(x) dx +$$

$$+ \rho \int_{\mathbb{R}} \Big\{ \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{1-\rho^2}} \phi\Big(\frac{y-\rho x}{\sqrt{1-\rho^2}}\Big) dy}_{=1} \Big\} x^2 \phi(x) dx = 0 + \rho \int x^2 \phi(x) dx = \rho$$

3) $\operatorname{cov}(X+Y+1, X-Y-1) = \operatorname{cov}(X+Y, X-Y) =$

$= \operatorname{cov}(X,X) - \cancel{\operatorname{cov}(X,Y)} + \cancel{\operatorname{cov}(Y,X)} - \operatorname{cov}(Y,Y) =$

$= \operatorname{Var}X - \operatorname{Var}Y = 0$ (ισόνομες)

4) (α) Κατ' αρχάς: $F_{X_{1:n}}(t) = 1 - P(X_{1:n} > t) \overset{\text{ανεξ.}}{=} 1 - \prod_1^n P(X_i > t) =$

$= 1 - \prod_{i=1}^n [1 - F_{X_i}(t)] \underset{\text{ισόνομες}}{=} 1 - [1-F(t)]^n \Rightarrow$

$\Rightarrow F_{X_{1:n}}(t) \underset{X \sim \mathcal{U}(0,1)}{=} 1-(1-t)^n, \ t \in (0,1).$

$\Rightarrow f_{X_{1:n}}(t) = n(1-t)^{n-1} = \dfrac{\Gamma(1+n)}{\Gamma(1)\Gamma(n)} t^{1-1}(1-t)^{n-1} 1(0<t<1)$

δηλ. $X_{1:n} \sim \mathcal{B}eta(1,n)$.

Αναλόγως: $F_{X_{n:n}}(t) = P(X_{n:n} \le t) = \prod_1^n P(X_i \le t) = \prod_{i=1}^n F_{X_i}(t) =$

$\underset{\text{ισόνομες}}{=} F(t)^n \underset{\mathcal{U}(0,1)}{=} t^n, \ t \in (0,1).$

$\Rightarrow f_{X_{n:n}}(t) = n t^{n-1} = \dfrac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} t^{n-1}(1-t)^{n-n} 1(0<t<1)$,

δηλ. $X_{n:n} \sim \mathcal{B}eta(n,1)$.

(β) $f_{X_{1:n}, X_{n:n}}(t,s) = n(n-1)(s-t)^{n-2} 1(0<t<s<1)$

(γ) $f_{X_{1:n}}(t) = \int_{\mathbb{R}} n(n-1)(s-t)^{n-2} 1(0<t<s<1) ds$

$= n(n-1) t^{n-2+1} \int_1^{1/t} (w-1)^{n-2} dw \ 1(0<t<1)$ $\left(\text{όπου } w \equiv \frac{s}{t}\right)$

(όπου $\xi = w-1$)

$= n(n-1) t^{n-1} \int_0^{\frac{1}{t}-1} \xi^{n-2} d\xi = n t^{n-1} \Big\{ \xi^{n-1} \Big|_0^{\frac{1-t}{t}} \Big\} =$

$= n(1-t)^{n-1}, \ t \in (0,1).$

5) (α) $F_{X_{1:n}, X_{n:n}}(x,y) = P(X_{1:n} \le x, X_{n:n} \le y) =$

$= P(X_{1:n} \le x) - P(X_{1:n} > x, X_{n:n} \le y)$

$= F_{X_{1:n}}(x) - P(x < X_i \le y \ , i = 1, 2, \dots, n) =$

$\overset{\text{ανεξ.}}{=} F_{X_{1:n}}(x) - \prod_{i=1}^{n} P(x < X_i \le y) =$

$= \left\{1 - \prod_{i=1}^{n} [1 - F(x)]\right\} - \prod_{i=1}^{n} [F(y) - F(x)]$

$= 1 - [1 - F(x)]^n - [F(y) - F(x)]^n =$

Ζισόν.

$\underset{(\text{ομοίως})}{=} F(y)^n - [F(y) - F(x)]^n \quad \forall x, y \in \mathbb{R} : x < y.$

(β) $f_{X_{1:n}, X_{n:n}}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X_{1:n}, X_{n:n}}(x,y) = \frac{\partial}{\partial x}\left\{ nF(y)^{n-1} f(y) - n[F(y) - F(x)]^{n-1} f(y) \right\}$

$= n(n-1)[F(y) - F(x)]^{n-2} f(x) f(y) \quad , x < y.$

(γ) $f_{X+Y}(z) = \int_{\mathbb{R}} f_Y(z-x) f_X(x)\, dx = \int_{\mathbb{R}} \lambda e^{-\lambda(z-x)} \mathbb{1}(z - x > 0)\, \lambda e^{-\lambda x} \mathbb{1}(x > 0)\, dx$

$= \int_{-\infty}^{+\infty} \lambda^2 e^{-\lambda z} \mathbb{1}(z > x > 0)\, dx = \lambda^2 e^{-\lambda z} \int_0^z dx =$

$= \lambda^2 z e^{-\lambda z} = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-\lambda z} \quad \forall z \in \mathbb{R}_+$

$\Rightarrow X + Y \sim \mathcal{G}(2, \lambda).$

Εναλλακτικά: $M_{X+Y}(t) = M_X(t) M_Y(t) = \left(1 + \frac{t}{\lambda}\right)^{-1} \left(1 + \frac{t}{\lambda}\right)^{-1} = \left(1 + \frac{t}{\lambda}\right)^{-2}$

$\Rightarrow X + Y \sim \mathcal{G}(2, \lambda).$