M 1226
Πιθανότητες ΙΙ
Β. Κουνιάς

Λύσεις 8

29/11/12

1) Για όλες τις περιπτώσεις έχουμε: $M_{\sum_1^n X_i}(t) = E\{e^{-t\sum_1^n X_i}\} =$

$$= E\Big\{\prod_1^n e^{-tX_i}\Big\} \overset{\text{ανεξ.}}{=} \prod_{i=1}^n E\{e^{-tX_i}\} = \prod_{i=1}^n M_{X_i}(t)$$

$$= M_{X_1}(t)^n \quad \text{αν και ισόνομες.}$$

Οπότε:

(α) $M_{S_n}(t) = \prod_{i=1}^n e^{-\mu_i t + \frac{1}{2}\sigma_i^2 t^2} = e^{-(\sum_1^n \mu_i)t + \frac{1}{2}(\sum_1^n \sigma_i^2)t^2}$
$\Rightarrow S_n \sim N(\sum_1^n \mu_i, \sum_1^n \sigma_i^2)$

(β) $M_{S_n}(t) = \prod_{i=1}^n \left(1+\frac{t}{\lambda}\right)^{-\alpha_i} = \left(1+\frac{t}{\lambda}\right)^{-\sum_1^n \alpha_i} \Rightarrow S_n \sim \Gamma(\sum_1^n \alpha_i, \lambda)$

(γ) $X_i \sim \chi^2_{n_i} \overset{d}{=} \Gamma\left(\frac{n_i}{2}, \frac{1}{2}\right) \overset{(\beta)}{\Rightarrow} S_n \sim \Gamma\left(\sum_1^n \frac{n_i}{2}, \frac{1}{2}\right) \overset{d}{=} \Gamma\left(\frac{\sum_1^n n_i}{2}, \frac{1}{2}\right) = \chi^2_{\sum_1^n n_i}$

(δ) $M_{S_n}(t) = \prod_1^n \exp\{-\lambda_i(1-e^{-t})\} = \exp\{-(\sum_1^n \lambda_i)(1-e^{-t})\} \Rightarrow$
$\Rightarrow S_n \sim Poisson(\sum_1^n \lambda_i)$

(ε) $M_{S_n}(t) = \prod_{i=1}^n (pe^{-t}+(1-p))^{n_i} = (pe^{-t}+(1-p))^{\sum_1^n n_i} \Rightarrow S_n \sim Bi(\sum_1^n n_i, p)$

(ζ) $M_{S_n}(t) = M_{X_1}(t)^n = (pe^{-t}+(1-p))^n \Rightarrow S_n \sim Bi(n,p)$
(ή από το (ε), με $n_i = 1 \ \forall i$)

(η) $M_{S_n}(t) = \prod_{i=1}^n \left(\frac{pe^{-t}}{1-(1-p)e^{-t}}\right)^{k_i} =$
$= \left(\frac{pe^{-t}}{1-(1-p)e^{-t}}\right)^{\sum_1^n k_i} \Rightarrow S_n \sim NBi(\sum_1^n k_i, p)$

(θ) $\text{Γεωμ.}(p) \overset{d}{=} NBi(1,p) \overset{(\eta)}{\Rightarrow} S_n \sim NBi(n,p)$

(ι) $\text{Εκθ.}(\lambda) \overset{d}{=} \Gamma(1,\lambda) \overset{(\beta)}{\Rightarrow} S_n \sim \Gamma(n,\lambda)$
$\left(\Rightarrow 2\lambda S_n \sim \Gamma(n, \tfrac{1}{2}) \overset{d}{=} \chi^2_{2n}.\right)$

2)(α) $f_{X+Y}(z) = \int_{\mathbb{R}} f_Y(z-x) f_X(x)\,dx = \lambda\mu\, e^{-\mu z}\int_{\mathbb{R}} e^{(\mu-\lambda)x}\,\mathbb{1}(z-x>0)\,\mathbb{1}(x>0)\,dx$

$$= \lambda\mu\, e^{-\mu z}\int_0^z e^{(\mu-\lambda)x}\,dx \underset{w\equiv(\mu-\lambda)x}{=} \frac{\lambda\mu}{\mu-\lambda}\, e^{-\mu z}\int_0^{(\mu-\lambda)z} e^{w}\,dw = \begin{cases}\lambda^2 z\, e^{-\lambda z} & \text{αν } \lambda=\mu \\ \frac{\lambda\mu}{\mu-\lambda}\, e^{-\mu z}\{e^{(\mu-\lambda)z}-1\} & \text{αν } \lambda\neq\mu\end{cases} =$$

$$= \begin{cases}\lambda^2 z\, e^{-\lambda z} & \text{αν } \lambda=\mu \\ \frac{\lambda\mu}{\mu-\lambda}\left[e^{-\lambda z}-e^{-\mu z}\right] & \text{αν } \lambda\neq\mu\end{cases}$$

(β) Όπως στην Α7.4(α), έχουμε: $1-F_{X\wedge Y}(t) = [1-F_X(t)][1-F_Y(t)] =$
$= e^{-\lambda t}e^{-\mu t} = e^{-(\lambda+\mu)t}$
$\Rightarrow X\wedge Y \sim$ Εκθ.$(\lambda+\mu)$.

3)(α) $\left.\begin{aligned} Z &\equiv \frac{X}{X+Y} \\ W &\equiv X+Y \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} X &= ZW \\ Y &= (1-Z)W \end{aligned}\right\} \Rightarrow \det J = \det\begin{pmatrix} w & z \\ -w & (1-z)\end{pmatrix}$

(βοηθητική)

$= (1-z)w + wz = w$

Άρα: $f_{Z,W}(z,w) = f_{X,Y}(zw, (1-z)w)\,|w| =$

$$= |w|\,\frac{\lambda^\alpha}{\Gamma(\alpha)}(zw)^{\alpha-1}e^{-\lambda zw}\,\mathbb{1}(zw>0)\,\frac{\lambda^\beta}{\Gamma(\beta)}((1-z)w)^{\beta-1}e^{-\lambda(1-z)w}\,\mathbb{1}((1-z)w>0)$$

$$= \mathbb{1}(w>0)\,\mathbb{1}(0<z<1)\,\frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)}\, w^{\alpha+\beta-1}e^{-\lambda w}\, z^{\alpha-1}(1-z)^{\beta-1}$$

$$\Rightarrow f_Z(z) = \int_{\mathbb{R}} f_{Z,W}(z,w)\,dw \underset{\xi=\lambda w}{=} \mathbb{1}(0<z<1)\, z^{\alpha-1}(1-z)^{\beta-1}\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\underbrace{\int_0^\infty \xi^{\alpha+\beta-1}e^{-\xi}\,d\xi}_{\Gamma(\alpha+\beta)}$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, z^{\alpha-1}(1-z)^{\beta-1}\,\mathbb{1}(0<z<1)$$

$\Rightarrow Z \sim$ Beta(α,β).

(β) Παρατηρούμε ότι:
$$f_{Z,W}(z,w) = f_Z(z)\cdot f_W(w).$$