M1226

Πιθανότητες ΙΙ

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Λύσεις 3

18/10/12

1) (α) $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}dx=\int_{-\infty}^{+\infty}\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)dx \overset{z:=(x-\mu)/\sigma}{=} \int_{-\infty}^{+\infty}\phi(z)\,dz=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-z^2/2}dz \overset{w:=z/\sqrt{2}}{=} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-w^2}dw=1$

Άσκ 1.4

βλ. 4.5, σ. 73 στις σημ.

(β) $P(\mu-k\sigma\le X\le\mu+k\sigma)=P\left(-k\le Z:=\frac{X-\mu}{\sigma}\le k\right)=$

$=\Phi(k)-\Phi(-k)=\Phi(k)-[1-\Phi(k))=2\Phi(k)-1=$

($\Phi'=\phi$ άρτια)

βλ. Πίνακα σ. 77

$=\begin{cases}2\cdot(0{,}8413)-1 & \text{για } k=1\\ 2\cdot(0{,}9332)-1 & \text{για } k=1{,}5\\ 2\cdot(0{,}9938)-1 & \text{για } k=2{,}5\end{cases}$

2) (α) $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\pi\beta}\,\frac{1}{1+\left(\frac{x-\alpha}{\beta}\right)^2}dx \overset{z:=\frac{x-\alpha}{\beta}}{=} \int_{-\infty}^{+\infty}\frac{1}{\pi(1+z^2)}dz=1$, από Άσκ. 1.5.

(β) $\displaystyle\int_{-\infty}^{+\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}1(x>0)=\int_0^{\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}dx \overset{z=\lambda x}{=}$

$\displaystyle=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}z^{\alpha-1}e^{-z}dz=1$ (από τον ορισμό της συνάρτησης $\Gamma(\alpha)$, $\alpha>0$)

(β) $\displaystyle F(x)=\int_{-\infty}^{x}\frac{1}{\pi\beta}\,\frac{1}{1+\left(\frac{z-\alpha}{\beta}\right)^2}dz \overset{y=(z-\alpha)/\beta}{=} \int_{-\infty}^{\frac{x-\alpha}{\beta}}\frac{1}{\pi}\,\frac{1}{1+y^2}dy=$

$=F_0\left(\frac{x-\alpha}{\beta}\right)$, $x\in\mathbb{R}$,

όπου: $\displaystyle F_0(z):=\int_{-\infty}^{z}\frac{1}{\pi}\,\frac{1}{1+y^2}dy \overset{y=\varepsilon\varphi w}{=}$

$\displaystyle=\int_{-\pi/2}^{\text{τοξεφ}(z)}\frac{1}{\pi}\,\cancel{\frac{1}{1+\varepsilon\varphi^2 w}}\,\cancel{\frac{1}{\sigma\upsilon\nu^2 w}}\,dw=\frac{1}{\pi}\left\{\text{τοξεφ}(z)+\frac{\pi}{2}\right\}=$

$=\frac{1}{2}+\frac{1}{\pi}\,\text{τοξεφ}(z)$, $z\in\mathbb{R}$.

Δηλαδή,

$F(x)=F_0\left(\frac{x-\alpha}{\beta}\right)=\frac{1}{2}+\frac{1}{\pi}\,\text{τοξεφ}\left(\frac{x-\alpha}{\beta}\right)$, $x\in\mathbb{R}$,

με $\alpha\in\mathbb{R}$, $\beta>0$, εφόσον $X\sim \text{Cauchy}(\alpha,\beta)$.

Έστω τώρα $X \sim \text{Εκθ.}(\lambda) \overset{d}{\equiv} \text{Γάμμα}(1,\lambda)$:

$$F(x) = \int_{-\infty}^{x} \lambda e^{-\lambda z} 1(z>0)\, dz = \int_{0}^{x} \lambda e^{-\lambda z} dz = \left(-e^{-\lambda z}\right)_0^x =$$
$$= 1 - e^{-\lambda x}$$ εφόσον $x \geq 0$ και $F(x)=0$ για $x<0$

Δηλ., αν $X \sim \text{Εκθ.}(\lambda)$, τότε $F(x) = \left[1-e^{-\lambda x}\right] 1(x>0)$, $x \in \mathbb{R}$.
$$\Rightarrow P(X>x) = 1-F(x) = e^{-\lambda x} \quad \forall x>0.$$

Αν τώρα $X \sim \Gamma(2,\lambda)$, δηλ., $f(x) = \frac{\lambda^2}{\Gamma(2)} x^{2-1} e^{-\lambda x} 1(x>0)$, $x \in \mathbb{R}$
$$= \lambda^2 x e^{-\lambda x} 1(x>0)$$

$$\Rightarrow F(x) = \int_{-\infty}^{x} f(y)dy = \int_0^x \lambda^2 y e^{-\lambda y} dx \underset{z=\lambda y}{=} \int_0^{\lambda x} z e^{-z} dz =$$
$$= \left(-ze^{-z}\right)_0^{\lambda x} + \int_0^{\lambda x} e^{-z} dz = -\lambda x e^{-\lambda x} + 1 - e^{-\lambda x}$$ (από ανωτέρω)
$$\Rightarrow F(x) = 1 - (1+\lambda x) e^{-\lambda x}$$, για $x>0$ και 0 για $x \leq 0$.

Σημ. Για $\alpha \in \mathbb{N}$ η σ.κ. της $\Gamma(\alpha,\lambda)$ υπολογίζεται αναλόγως, ενώ για $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ δεν υπάρχει "κλειστή" μορφή της.

(γ) Αν $X \sim \text{Εκθ.}(\lambda)$: $P(X \leq \frac{1}{\lambda}) = F(\frac{1}{\lambda}) = 1 - e^{-\lambda \cdot \frac{1}{\lambda}} = 1 - e^{-1}$

$$P\left(\frac{1}{\lambda} \leq X \leq \frac{2}{\lambda}\right) = F\left(\frac{2}{\lambda}\right) - F\left(\frac{1}{\lambda}\right) = (1-e^{-2}) - (1-e^{-1}) = e^{-1} - e^{-2}$$

$$P\left(X \leq \frac{2}{\lambda} \middle| X > \frac{1}{\lambda}\right) = \frac{P\left(\frac{1}{\lambda} < X \leq \frac{2}{\lambda}\right)}{P(X>1/\lambda)} = \frac{F(\frac{2}{\lambda}) - F(\frac{1}{\lambda})}{1-F(1/\lambda)} =$$
$$= \frac{e^{-1}-e^{-2}}{e^{-1}} = 1 - e^{-1} = P\left(X \leq \frac{1}{\lambda}\right) (!)$$

3) (α) $F_Y(y) = P(Y \leq y) = P(X^{1/\alpha} \leq y) = P(X \leq y^\alpha) = \begin{cases} 1 - e^{-\lambda y^\alpha} & \forall y>0 \\ 0 & \forall y \leq 0 \end{cases}$

$\Rightarrow f_Y(y) = \alpha \lambda y^{\alpha-1} e^{-\lambda y^\alpha} 1(y>0)$, δηλ., $Y \sim \text{Weibull}(\alpha,\lambda)$

Αλλιώς: $g(x) := x^{1/\alpha}, x>0 \Rightarrow g^{-1}(y) = y^\alpha, y>0 \Rightarrow$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \lambda e^{-\lambda y^\alpha} \alpha y^{\alpha-1} = \alpha \lambda y^{\alpha-1} e^{-\lambda y^\alpha}, \quad y>0.$$

(β) Έστω τώρα $Y \sim Weib(\alpha,\lambda)$: $F_Y(y) = \int_{-\infty}^{y} \alpha\lambda\psi^{\alpha-1}e^{-\lambda\psi^\alpha}\mathbb{1}(\psi>0)\,d\psi$

$$= \int_0^y \alpha\lambda\psi^{\alpha-1}e^{-\lambda\psi^\alpha}d\psi \overset{z=\psi^\alpha}{=\!=} \int_0^{y^\alpha}\lambda e^{-\lambda z}dz = \begin{cases}1-e^{-\lambda y^\alpha}, & y\geq 0\\ 0, & y<0\end{cases}$$

$$\Rightarrow P\left(Y>\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right) = \cancel{1-\left\{1-e^{-\lambda\left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right)^\alpha}\right\}}$$

4) Έστω $X \sim \mathcal{G}(\alpha,\lambda)$, $Y := cX$, $c>0$. Τότε:

$$f_Y(y) = f_X(y/c)\left|\frac{d}{dy}y/c\right| = \frac{1}{c}\frac{\lambda^\alpha}{\Gamma(\alpha)}(y/c)^{\alpha-1}e^{-\lambda y/c} =$$

$$= \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-(\lambda/c)y}, \quad y>0. \text{ Άρα: } Y\sim\mathcal{G}\left(\alpha,\frac{\lambda}{c}\right).$$

Οπότε εφόσον $X\sim\mathcal{G}(n,\lambda) \Rightarrow Y = 2\lambda X \sim \mathcal{G}\left(n,\frac{\lambda}{2\lambda}\right) \overset{d}{=}$

$$\overset{d}{=} \mathcal{G}\left(n,\frac{1}{2}\right) \overset{d}{=} \mathcal{G}\left(\frac{2n}{2},\frac{1}{2}\right) \overset{d}{=:} \chi^2_{2n}.$$

5) Εδώ, $X\sim\mathcal{U}(0,1)$, $Y := -\frac{1}{\lambda}\log(1-X)$. Οπότε:

Θέτοντας: $y = g(x) = -\frac{1}{\lambda}\log(1-x) \Rightarrow x = g^{-1}(y) = 1-e^{-\lambda y}$

$$\Rightarrow f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| = f_X(1-e^{-\lambda y})\left|+\lambda e^{-\lambda y}\right| =$$

$$= \mathbb{1}(0<1-e^{-\lambda y}<1)\,\lambda e^{-\lambda y}$$

$$= \mathbb{1}(0<e^{-\lambda y}<1)\,\lambda e^{-\lambda y} = \lambda e^{-\lambda y}\mathbb{1}(-\lambda y<0) =$$

$$= \lambda e^{-\lambda y}\mathbb{1}(y>0), \text{ δηλ., } Y\sim \mathcal{E}xp(\lambda).$$