M 2611 (1/2)

Π. Στατιστική

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Λύσεις 2

11/10/12

1)(α) $E_\theta \hat{\vartheta}_n = 2 E_\theta \bar{X}_n = 2 E_\theta X_1 = 2\,\frac{0+\theta}{2} = \theta \quad \forall \theta > 0$

$\Rightarrow MT\Sigma(\theta|\hat{\vartheta}_n) = Var_\theta \hat{\vartheta}_n = 4\, Var_\theta \bar{X}_n = 4\,\frac{Var_\theta X_1}{n} =$

$= \frac{4}{n}\,\frac{(\theta-0)^2}{12} = \theta^2/(3n)$

(β) Παρατηρούμε ότι $Y \equiv \frac{X}{\theta} \sim U(0,1)$ και $X_{nn} = \theta Y_{nn}$

με $Y_{nn} \sim Beta(n,1)$ (από Α.1.4)

$\Rightarrow E_\theta \hat{\theta}_n = \theta E Y_{nn} = \theta \frac{n}{n+1} \Rightarrow b(\theta|\hat{\theta}_n) = -\frac{\theta}{n+1} \quad \forall \theta > 0$

και $Var_\theta \hat{\theta}_n = \theta^2 Var Y_{nn} = \theta^2 \frac{n}{(n+1)^2(n+2)}$

$\Rightarrow MT\Sigma(\theta|\hat{\theta}_n) = Var_\theta \hat{\theta}_n + b^2(\theta|\hat{\theta}_n) = \frac{2\theta^2}{(n+1)(n+2)}$

[Παρατηρείστε (δεν ζητείται) ότι: $\bar{b}(\theta|\hat{\theta}_n) = \frac{-\theta/(n+1)}{\theta\sqrt{\frac{n}{n+2}}\,\frac{1}{n+1}} =$

$= -\sqrt{1+\frac{2}{n}} \xrightarrow[n\to\infty]{} -1 \neq 0$

δηλ., η $\hat{\theta}_n$ δεν είναι ούτε ασυμπτωτικά αμερόληπτη!]

<u>Όμως</u>: $MT\Sigma(\theta|\hat{\theta}_n) < MT\Sigma(\theta|\hat{\vartheta}_n) \; \forall \theta$ εφόσον $n \geq 3$.

(γ) $\hat{\vartheta}_n = 2\bar{X}_n \xrightarrow{P_\theta} 2 E_\theta X_1 = 2\,\frac{0+\theta}{2} = \theta \quad \forall \theta$ από τον ΑΝΜΑ,

δηλ. $\hat{\vartheta}_n$ συνεπής για την θ.

Επίσης όμως: $P_\theta(|\hat{\theta}_n - \theta| > \varepsilon) \leq \frac{MT\Sigma(\theta|\hat{\theta}_n)}{\varepsilon} =$

$= \frac{2\theta^2}{\varepsilon^2(n+1)(n+2)} \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$ και $\forall \theta \in \Theta$.

$\Rightarrow \hat{\theta}_n \xrightarrow{P_\theta} \theta$, δηλ., $\hat{\theta}_n$ συνεπής για την θ.

2) $T_n \equiv \sum_{i=1}^n X_i \sim Poisson(n\theta) \Rightarrow E_\theta \hat{\eta}_n =$

$= E_\theta\left\{\left(1-\frac{1}{n}\right)^{T_n}\right\} = \sum_{t=0}^\infty \left(1-\frac{1}{n}\right)^t e^{-n\theta}\frac{(n\theta)^t}{t!} =$

$$= e^{-n\theta} \sum_{t=0}^{\infty} \left[n\theta\left(1-\frac{1}{n}\right)\right]^t \Big/ t! = e^{-n\theta} e^{n\theta(1-\frac{1}{n})} = e^{-\theta} =$$

$$\Rightarrow \hat{\eta}_n \text{ αμερόληπτη για την } \eta \ ! \qquad \left(= P_\theta(X_1=0) = \eta \right.$$

Τώρα, $E_\theta \hat{\eta}_n^2 = E_\theta \left(1-\frac{1}{n}\right)^{2T_n} = \dots = e^{-n\theta} e^{n\theta(1-\frac{1}{n})^2} =$

$$= e^{-2\theta + \theta/n} \quad \forall \theta \Rightarrow$$

$$\text{ΜΤΣ}(\eta|\hat{\eta}_n) = \mathrm{Var}_\theta(\hat{\eta}_n) = \exp\left\{-2\theta + \frac{\theta}{n}\right\} - e^{-2\theta} = e^{-2\theta}\left\{e^{\frac{\theta}{n}} - 1\right\}$$

$$\xrightarrow[n\to\infty]{} 0 \Rightarrow \hat{\eta}_n \xrightarrow{P_\theta} \eta \ \forall \theta, \text{ δηλ., } \hat{\eta}_n \text{ συνεπής για την } \eta.$$

3) (α) $E_\lambda \hat{\hat{\lambda}}_n = \frac{\alpha}{n} n E_\lambda(X_1^{-1}) = \alpha \frac{\Gamma(\alpha-1)}{\Gamma(\alpha)\lambda^{-1}} = \frac{\alpha\lambda}{\alpha-1} \ \forall \lambda > 0.$

και $\mathrm{Var}_\lambda \hat{\hat{\lambda}}_n = \frac{\alpha^2}{n} \mathrm{Var}_\lambda(X_1^{-1}) = \frac{\alpha^2}{n}\left\{E(X_1^{-2}) - (E(X^{-1}))^2\right\}$

$$= \frac{1}{n}\left\{\alpha^2 \frac{\Gamma(\alpha-2)}{\Gamma(\alpha)\lambda^{-2}} - \frac{\alpha^2\lambda^2}{(\alpha-1)^2}\right\} = \frac{\alpha^2\lambda^2}{n}\left\{\frac{1}{(\alpha-1)(\alpha-2)} - \frac{1}{(\alpha-1)^2}\right\} =$$

$$= \frac{\alpha^2\lambda^2}{n(\alpha-1)^2(\alpha-2)} \quad \forall \lambda > 0$$

(β) Κατ' αρχάς: $T \equiv \Sigma_1^n X_i \sim \mathcal{G}(n\alpha, \lambda) \Rightarrow$

$$\Rightarrow E_\lambda \hat{\lambda}_n = n\alpha E T^{-1} = n\alpha \frac{\Gamma(n\alpha-1)}{\Gamma(n\alpha)\lambda^{-1}} = \frac{n\alpha\lambda}{n\alpha-1} \Rightarrow$$

$$\Rightarrow b(\lambda|\hat{\lambda}_n) = \frac{n\alpha\lambda}{n\alpha-1} - \lambda = \frac{\lambda}{n\alpha-1} \quad \forall \lambda > 0.$$

Επίσης: $E_\lambda \hat{\lambda}_n^2 = (n\alpha)^2 E T^{-2} = n^2\alpha^2 \frac{\Gamma(n\alpha-2)}{\Gamma(n\alpha)\lambda^{-2}} =$

$$= \frac{n^2\alpha^2\lambda^2}{(n\alpha-1)(n\alpha-2)} \Rightarrow \mathrm{Var}_\lambda \hat{\lambda}_n = \frac{n^2\alpha^2\lambda^2}{n\alpha-1}\left\{\frac{1}{n\alpha-2} - \frac{1}{n\alpha-1}\right\}$$

$$= \frac{n^2\alpha^2\lambda^2}{(n\alpha-1)^2(n\alpha-2)} \Rightarrow \bar{b}(\lambda|\hat{\lambda}_n) = \frac{\frac{\lambda}{n\alpha-1}}{\frac{n\alpha\lambda}{(n\alpha-1)\sqrt{n\alpha-2}}} =$$

$$= \frac{\sqrt{n\alpha-1}}{n\alpha} = \frac{1}{\sqrt{n\alpha}}\sqrt{1-\frac{1}{n\alpha}} \xrightarrow[n\to\infty]{} 0$$