M2611
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Λύσεις 3

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(α) $L(\lambda) = \prod_{i=1}^{n} \{2\lambda X_i e^{-\lambda X_i^2}\} = (2\lambda)^n (\prod_1^n X_i) e^{-\lambda \sum_1^n X_i^2}$

$\Rightarrow \ell(\lambda) = n \log\lambda + n\log 2 + \sum_1^n \log X_i - \lambda \sum_1^n X_i^2$

$\Rightarrow \ell'(\lambda) = \frac{n}{\lambda} - \sum_1^n X_i^2 = 0 \Leftrightarrow \tilde{\lambda} = \frac{n}{T}$, όπου $T = \sum_1^n X_i^2$

και $\ell''(\lambda) = -\frac{n}{\lambda^2} < 0 \ \forall \lambda$ $\Big\} \Rightarrow \hat{\lambda} = \frac{n}{T}$.

Τώρα, $Y_i \equiv X_i^2 \sim \mathcal{E}\kappa\theta(\lambda)$ (ελέγξτε το) και $X_1, \dots, X_n$ ανεξ.
$\Rightarrow T \sim \mathcal{G}(n, \lambda)$.

Οπότε: $E_\lambda \hat{\lambda} = n E_\lambda T^{-1} = n \frac{\Gamma(n-1)}{\Gamma(n)\lambda^{-1}} = \frac{n}{n-1}\lambda$

$\Rightarrow b(\lambda|\hat{\lambda}_n) = E_\lambda \hat{\lambda} - \lambda = \frac{\lambda}{n-1} \neq 0$, δηλ. $\hat{\lambda}$ όχι αμερόληπτη.

(β) Για $c = \frac{n-1}{n}$, $\hat{\theta}_n = c\hat{\lambda}_n = \frac{n-1}{T}$ είναι αμερόληπτη.

(γ) $E(\hat{\lambda})^2 = n^2 E T^{-2} = n^2 \frac{\Gamma(n-2)}{\Gamma(n)\lambda^{-2}} = \frac{n^2}{(n-1)(n-2)}\lambda^2$

$Var \hat{\lambda} = \frac{n^2}{(n-1)(n-2)}\lambda^2 - \frac{n^2}{(n-1)^2}\lambda^2 = \frac{n^2 \lambda^2}{(n-1)^2(n-2)}$

$\Rightarrow \bar{b}(\lambda|\hat{\lambda}) = \frac{b(\lambda|\hat{\lambda})}{\sqrt{Var_\lambda \hat{\lambda}}} = \frac{\cancel{\lambda/(n-1)}}{\cancel{\lambda/(n-1)}} \cdot \frac{\sqrt{n-2}}{n} \xrightarrow[n\to\infty]{} 0$

(δ) $P_\lambda(|\hat{\lambda} - \lambda| > \varepsilon) \leq \frac{MT\Sigma(\lambda|\hat{\lambda})}{\varepsilon^2} = \frac{Var_\lambda \hat{\lambda} + b^2(\lambda|\hat{\lambda})}{\varepsilon^2}$

$\xrightarrow[n\to\infty]{} 0 \ \forall \varepsilon > 0$ και $\forall \lambda \in \Theta = \mathbb{R}_+$

$\Rightarrow \hat{\lambda} \xrightarrow[n\to\infty]{P_\lambda} \lambda \ \forall \lambda$

(ε) $\frac{\partial}{\partial\lambda} \log f(x) = \frac{\partial}{\partial\lambda}\{\log 2 + \log\lambda + \log x - \lambda x^2\} = \frac{1}{\lambda} - x^2$

$\Rightarrow I_1(\lambda) = Var_\lambda\left\{\frac{\partial}{\partial\lambda}\log f(X_1|\lambda)\right\} = Var_\lambda\left\{\frac{1}{\lambda} - X_1^2\right\} = Var_\lambda(X_1^2) =$

$= Var_\lambda Y_1 = \frac{1}{\lambda^2}.$

Οπότε, το κάτω φράγμα της C-F-R για την διασπορά της αμερόληπτης $\hat{\vartheta}_n$ είναι: $B_n(\lambda) = \frac{1}{I_n(\lambda)} = \frac{1}{nI_1(\lambda)} = \frac{\lambda^2}{n}$ $\forall\lambda$,

ενώ για την διασπορά της $\hat{\lambda}_n$ είναι: $B'_n(\lambda) = \frac{\left(\frac{\partial}{\partial\lambda}E_\lambda\hat{\lambda}_n\right)^2}{I_n(\lambda)} =$

$= \left(\frac{n}{n-1}\right)^2\frac{\lambda^2}{n} < \left(\frac{n}{n-1}\right)^2\frac{\lambda^2}{n-2} = Var_\lambda\hat{\lambda}_n$ $\forall\lambda$.

Αλλά επίσης: $Var_\lambda\hat{\vartheta}_n = c^2 Var_\lambda\hat{\lambda}_n = \left(\frac{n-1}{n}\right)^2\cdot\left(\frac{n}{n-1}\right)^2\frac{\lambda^2}{n-2} = \frac{\lambda^2}{n-2} >$

$> B_n(\lambda) = \frac{\lambda^2}{n}$ $\forall\lambda$.

(ζ) Οπότε: $e(\lambda|\hat{\vartheta}_n) = \frac{B_n(\lambda)}{Var_\lambda\hat{\vartheta}_n} = \frac{\lambda^2/n}{\lambda^2/(n-2)} = \frac{n-2}{n} = 1 - \frac{2}{n} < 1$ (αλλά κοντά!)

και $a.e.(\lambda|\hat{\lambda}_n) = \lim\frac{B'_n(\lambda)}{Var_\lambda\hat{\lambda}_n} = \lim\frac{\left(\frac{n}{n-1}\right)^2\lambda^2/n}{\left(\frac{n}{n-1}\right)^2\frac{\lambda^2}{n-2}} = \lim\frac{n-2}{n} = \lim\left(1-\frac{2}{n}\right) = 1.$

(ασυμπτωτ. αμεροληψία)

(η) $\hat{\lambda} = \frac{n}{T} = \frac{1}{\bar{Y}}$, όπου $Y_i = X_i^2 \sim$ Εκθ.(λ), $\mu := \frac{1}{\lambda}$

$\sqrt{n}(\hat{\lambda}-\lambda) = \sqrt{n}\left(\frac{1}{\bar{Y}} - \frac{1}{\mu}\right) = \sqrt{n}(g(\bar{Y}) - g(\mu))$ (όπου $g(x) = 1/x$)

Αλλά, από ΚΟΘ $\sqrt{n}(\bar{Y}-\mu) \xrightarrow{d} N(0, Var Y_1) \overset{d}{=} N\left(0, \frac{1}{\lambda^2}\right)$,

και άρα, από το Θ. σταθεροποίησης της διασποράς, έχω

$\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} N(0, \underbrace{[g'(\mu)]^2 Var Y_1}_{\left(-\frac{1}{\mu^2}\right)^2\cdot\frac{1}{\lambda^2} = \lambda^4/\lambda^2 = \lambda^2}) \overset{d}{=} N(0, \lambda^2)$

$\lambda^2 \left(= \frac{1}{I_1(\lambda)}\right)$, $= c^2(\lambda)$

θ) $\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} N(0, \lambda^2)$

$g?$: $\sqrt{n}(g(\hat{\lambda}) - g(\lambda)) \xrightarrow{d} N(0, \underbrace{[g'(\lambda)]^2\lambda^2}_{=1})$ (από Θ. Στ. Διασπ.)

θέλουμε

$\Rightarrow g'(\lambda) = \pm\frac{1}{\lambda} \Rightarrow g(\lambda) = \pm\log\lambda$ $\forall\lambda$.

Δηλ., $\sqrt{n}(\log\hat{\lambda} - \log\lambda) \xrightarrow{d} N(0,1)$. (Με το "−" τι γίνεται;)