M 1226
Πιθανότητες ΙΙ
Β. Κλωνιάς

Λύσεις 4

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1) (α) $EX^4 = \dfrac{\Gamma(\alpha+4)}{\Gamma(\alpha)\lambda^4} = \dfrac{\Gamma((\alpha+3)+1)}{\Gamma(\alpha)\lambda^4} = \dfrac{(\alpha+3)\Gamma(\alpha+3)}{\Gamma(\alpha)\lambda^4} = \dfrac{(\alpha+3)(\alpha+2)(\alpha+1)\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\lambda^4}$

$= \dfrac{(\alpha+1)(\alpha+2)(\alpha+3)}{\lambda^4}$

(β) $EX^{-1} = \dfrac{\Gamma(\alpha-1)}{\Gamma(\alpha)\lambda^{-1}} = \dfrac{\Gamma(\alpha-1)\lambda}{\Gamma((\alpha-1)+1)} = \dfrac{\cancel{\Gamma(\alpha-1)}\lambda}{(\alpha-1)\cancel{\Gamma(\alpha-1)}} = \dfrac{\lambda}{\alpha-1}$

$EX^{-2} = \dfrac{\Gamma(\alpha-2)}{\Gamma(\alpha)\lambda^{-2}} = \dfrac{\cancel{\Gamma(\alpha-2)}\lambda^2}{(\alpha-1)(\alpha-2)\cancel{\Gamma(\alpha-2)}} = \dfrac{\lambda^2}{(\alpha-1)(\alpha-2)}$

$\Rightarrow Var(X^{-1}) = E(X^{-1})^2 - (EX^{-1})^2 = E(X^{-2}) - (E(X^{-1}))^2 =$

$= \dfrac{\lambda^2}{(\alpha-1)(\alpha-2)} - \dfrac{\lambda^2}{(\alpha-1)^2} = \dfrac{\lambda^2}{(\alpha-1)^2(\alpha-2)}$

(γ) Αν $Z \sim N(0,1)$, τότε η $Y := Z^2 \sim F_Y(y) = P(Z^2 \le y) =$

$= P(|Z| \le \sqrt{y}) = P(-\sqrt{y} \le Z \le \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) =$

$= \Phi(\sqrt{y}) - [1-\Phi(z)] = 2\Phi(\sqrt{y}) - 1 \Rightarrow$

$\Rightarrow f_Y(y) = 2\Phi'(\sqrt{y})\dfrac{1}{2\sqrt{y}} = \dfrac{1}{\sqrt{y}}\phi(\sqrt{y}) = \dfrac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2}(\sqrt{y})^2}$

$= \dfrac{(1/2)^{1/2}}{\Gamma(1/2)} y^{1/2-1} e^{-\frac{1}{2}y} \quad \forall y>0 \Rightarrow Y \equiv Z^2 \sim \mathcal{G}\left(\frac{1}{2},\frac{1}{2}\right)$

$\Rightarrow E(Z^{2k}) = E(Y^k) = \dfrac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{1}{2})(1/2)^k} = \dfrac{2^k}{\sqrt{\pi}}\Gamma\left(k+\frac{1}{2}\right)$

2) (α) $EX = \sum\limits_{x=k}^{\infty} x\binom{x-1}{k-1} p^k (1-p)^{x-k} =$

$= \sum\limits_{x=k}^{\infty} x \dfrac{(x-1)!}{(k-1)!(x-k)!} p^k (1-p)^{x-k}$

$= \dfrac{k}{p} \sum\limits_{x=k}^{\infty} \dfrac{x!}{k!(x-k)!} p^{k+1}(1-p)^{x-k} = \dfrac{k}{p} \sum\limits_{x^*=k^*}^{\infty} \binom{x^*-1}{k^*-1} p^{k^*}(1-p)^{x^*-k^*} =$

$k^* = k+1$
$x^* = x+1$

$= \dfrac{k}{p}$

(β) $EX^2 = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \frac{d^2}{dt^2}\left\{\frac{pe^{-t}}{1-(1-p)e^{-t}}\right\}^k\Bigg|_{t=0} = \left\{\frac{k\,(k+(1-p)e^{-t})\,M_X(t)}{(1-(1-p)\,e^{-t})^2}\right\}_{t=0} =$

$$= \frac{k(k+(1-p))}{(1-(1-p))^2} = \frac{k^2+k(1-p)}{p^2} = \frac{k(1-p)}{p^2} + \left(\frac{k}{p}\right)^2 \Rightarrow$$

$$\Rightarrow VarX = EX^2 - (EX)^2 = \frac{k(1-p)}{p^2} \cancel{+\left(\frac{k}{p}\right)^2} \cancel{-\left(\frac{k}{p}\right)^2}$$

3) $$EX = \int_{-\infty}^{+\infty} x\,\frac{\lambda}{2}e^{-\lambda|x-\theta|}\,dx = \theta + \int_{-\infty}^{+\infty}(x-\theta)\frac{\lambda}{2}e^{-\lambda|x-\theta|}\,dx$$

$$\overset{z:=(x-\theta)}{=} \theta + \underbrace{\int_{-\infty}^{+\infty} z\,\frac{\lambda}{2}\,e^{-\lambda|z|}\,dz}_{\text{περιττη}} = \theta + 0 = \theta$$

$$VarX = E(X-EX)^2 = E(X-\theta)^2 = \int_{-\infty}^{+\infty}(x-\theta)^2\frac{\lambda}{2}e^{-\lambda|x-\theta|}\,dx$$

$$= \int_{-\infty}^{+\infty}\underbrace{z^2\frac{\lambda}{2}\,e^{-|z|}}_{\text{άρτια}}\,dz = \cancel{2}\int_0^{\infty} z^2\,\frac{\lambda}{\cancel{2}}\,e^{-\lambda|z|}\,dz$$

$$= \int_0^{\infty} z^2\,\lambda\,e^{-\lambda z}\,dz =$$

$$\overset{y=\lambda z}{=} \frac{1}{\lambda^2}\int_0^{\infty} y^2 e^{-y}\,dy = \frac{1}{\lambda^2}\,\Gamma(3) = \frac{2}{\lambda^2}.$$