M2611

Π. Στατιστική

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Λύσεις 5

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Θα βρούμε τις επαρκείς στατιστικές T, των ασκήσεων 1-5, εφαρμόζοντας το Θεώρημα παραγοντοποίησης:

1) $$\prod_{i=1}^{n} p(X_i|\theta) = \prod_{i=1}^{n} \binom{X_i-1}{\kappa-1} \theta^{\kappa} (1-\theta)^{X_i-\kappa} = \underbrace{\theta^{n\kappa}(1-\theta)^{\sum_i^n X_i - n\kappa}}_{g(\theta, T=\sum_i^n X_i)} \cdot \underbrace{\prod_{i=1}^{n}\binom{X_i-1}{\kappa-1}}_{h(\underline{X})}.$$

2) $$\prod_{i=1}^{n} f(X_i|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} X_i^{\alpha-1} e^{-\lambda X_i} 1(X_i>0) =$$

$$= \left\{\lambda^{n\alpha} e^{-\lambda \sum_i^n X_i}\right\} \cdot \left\{\frac{1}{\Gamma(\alpha)^n}\left(\prod_1^n X_i\right)^{\alpha-1} 1(X_{1n}>0)\right\} \Rightarrow T = \sum_1^n X_i$$

3) $$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{1-\theta} 1(\theta<x_i<1) = \left\{\frac{1}{(1-\theta)^n} 1(x_{1n}>\theta)\right\} \cdot 1(x_{nn}<1)$$

$\Rightarrow T = X_{1n}$.

4) $$\prod_{i=1}^{n} f(X_i|\theta) = \prod_1 e^{-(X_i-\theta)} 1(X_i>\theta) = \underbrace{\left\{e^{n\theta} 1(X_{1n}>\theta)\right\}}_{g(\theta, T\equiv X_{1n})} \cdot \underbrace{e^{-\sum_1^n X_i}}_{h(\underline{X})}.$$

5) $$\prod_{i=1}^{n} f(X_i|\theta=(\lambda,\mu)) = \prod_{i=1}^{n} \lambda e^{-\lambda(X_i-\mu)} 1(X_i>\mu)$$

$$= \underbrace{\lambda^n e^{-\lambda\left(\sum_1^n X_i - n\mu\right)} 1(X_{1n}>\mu)}_{g\left(\theta=(\lambda,\mu),\ \underline{T}\equiv\left(\underbrace{\sum_1^n X_i}_{T_1},\ \underbrace{X_{1n}}_{T_2}\right)\right)} \cdot \underbrace{1}_{h(\underline{X})}$$

$\Rightarrow \underline{T} \equiv (T_1, T_2) \equiv \left(\sum_1^n X_i, X_{1n}\right)$ επαρκής για την $\underline{\theta} = (\lambda, \mu)$.

6) (α) Έστω $\mu(t) := E(U|T=t) = P(X_1=1 \mid \sum_1^n X_i = t) =$

$$= \frac{P(X_1=1, \sum_2^n X_i = t-1)}{P(\sum_1^n X_i = t)}$$, αλλά $\sum_1 X_i \sim NBi(n,p)$, οπότε:

$$\mu(t) = \frac{p \cdot p^{n-1}(1-p)^{t-1-(n-1)}\binom{(t-1)-1}{(n-1)-1}}{p^n(1-p)^{t-n}\binom{t-1}{n-1}} = \frac{\binom{t-2}{n-2}}{\binom{t-1}{n-1}} = \frac{n-1}{t-1}$$

$$\Rightarrow \hat{p} = d(\underline{X}) = \mu(T) = \frac{n-1}{T-1} = \frac{n-1}{\sum_1^n X_i - 1}.$$

Τώρα, $\operatorname{Var} U = P(X_1=1)(1-P(X_1=1)) = p(1-p)$.

Ενώ: $\operatorname{Var} d(\underline{X}) = E\, d(\underline{X})^2 - (E\, d(\underline{X}))^2 = E\, d(\underline{x})^2 - p^2$.

Αρκεί λοιπόν να διαπιστώσουμε ότι: $E(d(\underline{X})^2) \leq p$:

$$E\, d(X)^2 = (n-1)^2 E\frac{1}{(T-1)^2} = (n-1)^2 \sum_{t=n}^{\infty} \frac{1}{(t-1)^2}\binom{t-1}{n-1}p^n(1-p)^{t-n} =$$

$$= \sum_{t=n}^{\infty} \frac{n-1}{t-1}\binom{t-2}{n-2}p^n(1-p)^{t-n} \leq \sum_{t=n}^{\infty}\binom{t-2}{n-2}p^n(1-p)^{t-n} =$$

(διότι: $t \geq n \Rightarrow t-1 \geq n-1 \Rightarrow \frac{n-1}{t-1} \leq 1$)

$$= p\sum_{s=m}^{\infty}\binom{s-1}{m-1}p^m(1-p)^{s-m} = p.$$

($s \equiv t-1$, $m \equiv n-1$)

(β) $\mu(t) := E(U \mid T=t) = P(X_1=2 \mid \sum_1^n X_i = t) =$

$$= \frac{P(X_1=2, \sum_2^n X_i = t-2)}{P(\sum_1^n X_i = t)} = \frac{p(1-p)^{2-1}\binom{t-3}{n-2}p^{n-1}(1-p)^{t-n-1}}{\binom{t-1}{n-1}p^n(1-p)^{t-n}} =$$

$$= \frac{\binom{t-3}{n-2}}{\binom{t-1}{n-1}} = \frac{(n-1)(t-n)}{(t-1)(t-2)}$$

$$\Rightarrow d(\underline{X}) = \mu(T) = \frac{(n-1)(T-n)}{(T-1)(T-2)} \left(= \left(1-\frac{1}{n}\right)\frac{\bar{X}-1}{(\bar{X}-\frac{1}{n})(\bar{X}-\frac{2}{n})} \xrightarrow{P} \right.$$

$$\left. \xrightarrow[n\to\infty]{P} \frac{EX_1-1}{(EX_1)^2} = \frac{\frac{1}{p}-1}{(\frac{1}{p})^2} = p(1-p) \right)$$

(just checking!)