M1226

Πιθανότητες II

B. Μακρίδης

Λύσεις 5

1/11/12

1) $g(x) := e^{-x}$, οπότε $Y = g(X) \Rightarrow x = g^{-1}(y) = -\log y \Rightarrow$

$\Rightarrow f_Y(y) = f_X(-\log y)\left|-\frac{1}{y}\right| 1(y>0) = \frac{1}{y} e^{+\log y} 1(-\log y > 0) 1(y>0)$

$= 1(0<y<1) \Rightarrow Y \sim \mathcal{U}(0,1)$.

2) $f_Y(y) = f_X\left(\frac{1}{y}\right)\left|-\frac{1}{y^2}\right| = \frac{1}{\pi}\frac{1}{1+\left(\frac{1}{y}\right)^2}\cdot\frac{1}{y^2} = \frac{1}{\pi}\frac{1}{1+y^2}$, $y \in \mathbb{R}$

$\Rightarrow$ Αν $X \sim Cauchy(0,1) \Rightarrow Y = \frac{1}{X} \sim Cauchy(0,1)$.

3) $P(\mu - k\sigma \le X \le \mu + k\sigma) = P(-k\sigma \le X - \mu \le k\sigma) =$

$= P(|X-\mu| \le k\sigma) = 1 - P(|X-\mu| > k\sigma) \geq 1 - \frac{Var X}{k^2\sigma^2} =$

$= 1 - \frac{1}{k^2} = \begin{cases} 0 & \text{για } k=1 \\ 1/2 & \text{για } k=\sqrt{2} \\ 5/9 & \text{για } k=1{,}5 \\ \frac{21}{25} = 0{,}84 & \text{για } k=2{,}5 \end{cases}$

5) Αν θέσουμε $g(x) := e^{-tx}$, έχουμε $g''(x) = t^2 e^{-tx} > 0 \Rightarrow g$ κυρτή

$\underset{\text{για } t=1}{\Rightarrow} E e^{-X} \geq e^{-EX} \underset{\text{γενικά}}{\Rightarrow} \underbrace{E e^{-tX}}_{M_X(t)} \geq e^{-tEX}$.

6) Έστω $g(x) = \log x$ κοίλη $\Rightarrow E\log|Y| \le \log E|Y|$

$\underset{\text{για } Y \equiv \frac{X-\mu}{\sigma}}{\Rightarrow} 2E\log\left|\frac{X-\mu}{\sigma}\right| \le \log E\left|\frac{X-\mu}{\sigma}\right|^2 = \log\frac{Var X}{\sigma^2} = \log 1 = 0$.

7) Από Chebyshev: $P(|X-\mu| \le \varepsilon) \geq 1 - \frac{Var X}{\varepsilon^2}$ $\forall \varepsilon > 0$

Οπότε, αν $Var X = 0$, έχουμε $P(|X-EX| \le \varepsilon) = 1$ $\forall \varepsilon > 0$

$\Rightarrow P(|X-EX| = 0) = 1 \Rightarrow P(X = EX) = 1$.