M 1226

Πιθανότητες ΙΙ

Βικλωνιάς.

Λύσεις 6

15/11/2012

1) $$\int_{\mathbb{R}} f(x,y)\,dy = \frac{1}{2\pi}\frac{1}{(1+x^2)^{3/2}}\int_{-\infty}^{+\infty}\frac{1}{(1+z^2)^{3/2}}\,dz \quad \left(\text{όπου: } z = \frac{y}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{2\pi}\frac{1}{(1+x^2)}\, 2\int_0^{\pi/2}\frac{1}{(1+\varepsilon\varphi^2\omega)^{3/2}}\frac{1}{\sigma\upsilon\nu^2\omega}\,d\omega \quad (\text{όπου: } \omega = \tau o\xi\varepsilon\varphi\, z)$$

$$= \frac{1}{\pi}\frac{1}{1+x^2}\int_0^{\pi/2}\sigma\upsilon\nu\omega\,d\omega = \frac{1}{\pi}\frac{1}{1+x^2} \quad \left(\Rightarrow f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}\right)\ \forall x\in\mathbb{R}.$$

Άρα:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dx\,dy = \int_{\mathbb{R}}\frac{1}{\pi}\frac{1}{1+x^2}\,dx = 1$$

2) $$\int_{\mathbb{R}} f(x,y)\,dy = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}x^{\alpha-1}1(x>0)\int_{\mathbb{R}} y^{\beta-1}(1-x-y)^{\gamma-1}1(y>0)\,\cdot$$

$$\cdot\, 1(x+y<1)\,dy = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}x^{\alpha-1}1(x>0)(1-x)^{\beta+\gamma-1}\int_0^1 z^{\beta-1}(1-z)^{\gamma-1}\,dz\cdot 1(x<1)$$

$$\left(\text{όπου: } z = \frac{y}{1-x}\right) \quad = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}x^{\alpha-1}(1-x)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}1(0<x<1)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}x^{\alpha-1}(1-x)^{\beta+\gamma-1}1(0<x<1) \quad (\Rightarrow X\sim Beta(\alpha,\beta+\gamma))$$

$$\Rightarrow \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\,dx\,dy = \int_{\mathbb{R}} f_X(x)\,dx = 1.$$

3) $$p_{X+Y}(z) = P(X+Y=z) = \sum_{x\in\mathbb{N}_0} P(X+Y=z\mid X=x)P(X=x) =$$

$$= \sum_{x=0}^{\infty} P(Y=z-x\mid X=x)\,p_X(x) = \sum_{x=0}^{\infty} P(Y=z-x)\,p_X(x) =$$

$$= \sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^{z-x}}{(z-x)!}1(z-x\in\mathbb{N}_0)\frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-(\lambda+\mu)}}{z!}\sum_{x=0}^{z}\binom{z}{x}\lambda^x\mu^{z-x} =$$

$$= \frac{e^{-(\lambda+\mu)}}{z!}(\lambda+\mu)^z\ \forall z\in\mathbb{N}_0 \Rightarrow Z := X+Y\sim Poisson(\lambda+\mu). \Rightarrow$$

$$\Rightarrow P(X=x \mid X+Y=z) = \frac{P(X=x,\ X+Y=z)}{P(X+Y=z)} = p_{X+Y}(z)^{-1} P(X=x,\ Y=z-x) =$$

$$= p_{X+Y}(z)^{-1}\ P(X=x)\,P(Y=z-x)\ \mathbb{1}(x,\ z-x \in \mathbb{N}_0) =$$

$$= \frac{(e^{-\lambda}\lambda^x/x!)(e^{-\mu}\mu^{z-x}/(z-x)!)}{e^{-(\lambda+\mu)}(\lambda+\mu)^z/z!}\ \mathbb{1}(\mathbb{N}_0 \ni x \leq z \in \mathbb{N}_0)$$

$$= \binom{z}{x} p^x (1-p)^{z-x}\ \mathbb{1}(x \in \{0,1,\dots,z\})\ \mathbb{1}(z \in \mathbb{N}_0)$$, όπου $p = \frac{\lambda}{\lambda+\mu}$

Δηλ., $X \mid X+Y=z \sim \mathrm{Bi}(z,p)$. $\forall z \in \mathbb{N}_0$.