M1226 (1/4)

Πιθανότητες II **Λύσεις 1** 3/10/2012

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1) Αναπτύσσουμε κατά Taylor την $(1-q)^{-k}$ =

$$= (1-q)^{-k}\Big|_{q=0} + k(1-q)^{-k-1}\Big|_{q=0}\frac{q}{1!} + k(k+1)(1-q)^{-k-2}\Big|_{q=0}\frac{q^2}{2!} +$$

$$+ k(k+1)(k+2)(1-q)^{-k-3}\Big|_{q=0}\frac{q^3}{3!} + \cdots =$$

$$= 1 + kq + \frac{k(k+1)}{2!}q^2 + \frac{k(k+1)(k+2)}{3!}q^3 + \cdots =$$

$$= \sum_{j=k}^{\infty}\binom{j-1}{k-1}q^{j-k},$$ και θέτοντας $q:=1-p$ επαληθεύουμε το ότι η σ.μ.π. της NB_i είναι πράγματι σ.μ.π.

2) $$e^{\theta} = e^{\theta}\Big|_{\theta=0} + e^{\theta}\Big|_{\theta=0}\frac{\theta}{1!} + e^{-\theta}\Big|_{\theta=0}\frac{\theta^2}{2!} + \cdots$$

$$= 1 + \frac{\theta}{1!} + \frac{\theta^2}{2!} + \cdots = \sum_{j=0}^{\infty}\frac{\theta^j}{j!}.$$

3)(α) $$\sum_{j=3}^{\infty}(1-p)^{j-1} \underset{i:=j-}{=} (1-p)^2\sum_{j=3}^{\infty}(1-p)^{j-3} \underset{i=j-3}{=} (1-p)^2\sum_{i=0}^{\infty}(1-p)^i =$$

$$= (1-p)^2\frac{1}{1-(1-p)} = (1-p)^2/p.$$

(β) $$\sum_{j=3}^{\infty}\frac{\theta^j}{(j-1)!} = \theta\sum_{j=3}^{\infty}\frac{\theta^{j-1}}{(j-1)!} \underset{k=j-1}{=} \theta\sum_{k=2}^{\infty}\frac{\theta^k}{k!} =$$

$$= \theta\left(e^{\theta} - \frac{\theta^0}{0!} - \frac{\theta^1}{1!}\right) = \theta(e^{\theta} - 1 - \theta).$$

(γ) $$\sum_{j=3}^{\infty}(1-p)^{2j-1} = (1-p)\sum_{j=3}^{\infty}(1-p)^{2(j-1)} =$$

$$\underset{i=j-1}{=} (1-p)\sum_{i=2}^{\infty}\left[(1-p)^2\right]^i = (1-p)\left\{\frac{1}{1-(1-p)^2} - 1 - (1-p)^2\right\}$$

$$= (1-p)\left\{\frac{1}{p(2-p)} - 1 - (1-p)^2\right\}.$$

4) $\left(\int_{-\infty}^{+\infty} e^{-z^2}dz\right)^2 = 4\int_0^{\infty}\int_0^{+\infty} e^{-(z^2+y^2)}\,dz\,dy \overset{\left(\begin{smallmatrix}x=r\cos\theta\\ y=r\sin\theta\end{smallmatrix}\right)}{=}$

$=4\int_0^{\pi/2}\left(\int_0^{\infty} e^{-r^2}\, r\,dr\right)d\theta \underset{x=z^2}{=} 4\int_0^{\pi/2}d\theta\ \frac{1}{2}\int_0^{\infty} e^{-x}dx = \pi$

$\Rightarrow \int_{\mathbb{R}} e^{-z^2}dz=\sqrt{\pi} \Rightarrow \int_0^{\infty} z^{-1/2}e^{-z}dz \overset{w=\sqrt{z}}{=} \int_0^{\infty} \frac{1}{w} e^{-w^2}\,2w\,dw =$

$= 2\int_0^{\infty} e^{-w^2}dw = \int_{-\infty}^{+\infty} e^{-w^2}dw = \sqrt{\pi}.$

5) $\int_0^{\infty} \frac{1}{1+x^2}dx \underset{x=\varepsilon\varphi\omega}{=} \int_0^{\pi/2} \sigma\upsilon\nu^2\omega\cdot\frac{1}{\sigma\upsilon\nu^2\omega}\,d\omega = \frac{\pi}{2}$

6) $\forall z\in\mathbb{R}_+ \int_{\mathbb{R}} e^{-(z-x)}\, 1(z-x>0)\, e^{-x}\, 1(x>0)\,dx =$

$= \int_{\mathbb{R}} e^{-z}\, 1(0<x<z)\,dx = e^{-z}\int_0^{z} dx = z e^{-z}$

Επίσης,

$\forall z\in\mathbb{R} \int_{\mathbb{R}} e^{-(z+x)}\, 1(z+x>0)\, e^{-x}\, 1(x>0)\,dx =$

$= \int_{\mathbb{R}} e^{-z} e^{-2x}\, 1(x>\underset{z_{max}}{0\vee(-z)})\,dx =$

$= e^{-z}\int_{0\vee(-z)}^{\infty} e^{-2x}dx = \frac{1}{2}e^{-z}\left(-e^{-2x}\Big|_{0\vee(-z)}^{+\infty}\right) =$

$= \frac{1}{2}e^{-z}\, e^{-2(0\vee(-z))} = \frac{1}{2}\left\{\begin{matrix} e^{-z} & \text{άν } z\geq 0 \\ e^{-z}\cdot e^{2z} & \text{άν } z<0 \end{matrix}\right\} =$

$= \frac{1}{2}\left\{\begin{matrix} e^{-z} & \text{άν } z\geq 0 \\ e^{z} & \text{άν } z<0 \end{matrix}\right\} = \frac{1}{2}e^{-|z|}.$

7) $\int_0^{\infty}\int_0^{\infty} e^{-xy}\, e^{-x(y+1)}\, x\ dx\,dy = \int_0^{\infty} e^{-x}x\left(\int_0^{\infty} e^{-2xy}dy\right)dx$

$= \int_0^{\infty} e^{-x}\cancel{x}\,\frac{1}{2\cancel{x}}\,dx = \frac{1}{2}$

8) • $P(X=Y) = \sum_{y=1}^{\infty} P(X=Y\mid Y=y)\,P(Y=y) =$

τύπος ολικής πιθανότητας

$$= \sum_{y=1}^{\infty} P(X=y \mid Y=y)\,P(Y=y) = \underbrace{\sum_{y=1}^{\infty} P(X=y)\,P(Y=y)}_{\text{ανεξ. των } X, Y} =$$

$$= \sum_{y=1}^{\infty} p(1-p)^{y-1} q(1-q)^{y-1} = pq \sum_{y=0}^{\infty} \left[(1-p)(1-q)\right]^{y} =$$

$$= \frac{pq}{1-(1-p)(1-q)} \left(= \frac{p}{2-p} \text{ άν } p=q \right).$$

- $P(X<Y) = \sum_{x=1}^{\infty} P(Y>X \mid X=x)\,P(X=x) =$

$$= \sum_{x=1}^{\infty} P(Y>x \mid X=x)\,P(X=x) = \sum_{x=1}^{\infty} P(Y>x)\,P(X=x)$$

$$= \sum_{x=1}^{\infty} (1-q)^{x} \cdot p(1-p)^{x-1} = p(1-q) \sum_{x=0}^{\infty} \left[(1-p)(1-q)\right]^{x}$$

$$= \frac{p(1-q)}{1-(1-p)(1-q)}.$$

- $P(X \text{ άρτιος}) = \sum_{k=1}^{\infty} P(X=2k) = \sum_{k=1}^{\infty} p^{2k}(1-p)^{2k-1} =$

$$= p^2(1-p) \sum_{k=1}^{\infty} \left[p(1-p)\right]^{2(k-1)} = p^2(1-p) \sum_{k=0}^{\infty} \left(\left[p(1-p)\right]^2\right)^{k}$$

$$= \frac{p^2(1-p)}{1-p^2(1-p)^2}.$$

- $P(X \text{ περιττός} \mid Y \text{ άρτιος}) \underset{X, Y \text{ ανεξ.}}{=} P(X \text{ περιττός}) =$

$$= 1 - P(X \text{ άρτιος}) = 1 - \frac{p^2(1-p)}{1-p^2(1-p)^2}.$$

9) $P(X=k \mid X+Y=n)$ για $k=0,1,\dots,n$

$$= \frac{P(X=k,\, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k,\, Y=n-k)}{P(X+Y=n)} =$$

$$= \frac{P(X=k)\,P(Y=n-k)}{P(X+Y=n)} = \frac{e^{-\theta}\frac{\theta^{k}}{k!}\; e^{-\theta}\frac{\theta^{n-k}}{(n-k)!}}{e^{-2\theta}\frac{(2\theta)^{n}}{n!}} =$$

$$= \frac{n!}{k!\,(n-k)!}\left(\frac{1}{2}\right)^{n} = \binom{n}{k}\left(\frac{1}{2}\right)^{n}.$$

Όπου, έγινε χρήση του ότι: $X+Y \sim \text{Poisson}(2\theta)$ – κάτι που μπορεί να το δει κανείς άμεσα είτε με χρήση ροπογεννητριών είτε με τη συνέλιξη των δύο σ.μ.π. των X, Y:

$$P(X+Y=n) = \sum_{x=0}^{\infty} P(X+Y=n \mid X=x)\,P(X=x)$$

$$= \sum_{x=0}^{\infty} P(x+Y=n \mid X=x)\,P(X=x) =$$

$$= \sum_{x=0}^{\infty} P(Y=n-x)\,P(X=x) =$$

(ανεξ. των X, Y)

$$= \sum_{x=0}^{\infty} p_Y(n-x)\,p_X(x) =$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\theta}\theta^{n-x}}{(n-x)!}\mathbb{1}(n-x\in\mathbb{N}_0)\,\frac{e^{-\theta}\theta^{x}}{x!}\mathbb{1}(x\in\mathbb{N}_0)$$

$$= \frac{e^{-2\theta}\theta^{n}}{n!}\sum_{x=0}^{n}\binom{n}{x} = \frac{e^{-2\theta}\theta^{n}}{n!}(1+1)^{n} =$$

$$= \frac{e^{-2\theta}(2\theta)^{n}}{n!} \quad \forall\, n\in\mathbb{N}_0 .$$