M 2611
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Λύσεις I

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1)(α) (i) $M_{\sum_{i=1}^n X_i}(t) = E\, e^{-t\Sigma X_i} = E \prod_{i=1}^n e^{-tX_i} \underset{\text{ανεξ.}}{=} \prod_{i=1}^n E e^{-tX_i} = \prod_{i=1}^n M_{X_i}(t)$

Εδώ $M_{X_i}(t) = (1+\frac{t}{\lambda})^{-\alpha_i}$ και άρα $M_{\sum_1^n X_i}(t) = (1+\frac{t}{\lambda})^{-\sum_{i=1}^n \alpha_i}$

$\Rightarrow \sum_1^n X_i \sim \mathcal{G}(\sum_1^n \alpha_i, \lambda)$.

(ii) Αλλά επίσης: $f_{X_1+X_2}(z) = \int_{\mathbb{R}} f_{X_2}(z-x) f_{X_1}(x)\,dx =$

$= \int_{\mathbb{R}} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} (z-x)^{\alpha_2-1} e^{-\lambda(z-x)} \mathbb{1}(z-x>0) \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-\lambda x} \mathbb{1}(x>0)\,dx =$

$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \int_0^z (z-x)^{\alpha_2-1} x^{\alpha_1-1}\,dx =$

$= \text{»} \cdot e^{-\lambda z}\, z^{\alpha_1+\alpha_2-1} \int_0^1 w^{\alpha_1-1}(1-w)^{\alpha_2-1}\,dw$ (όπου: $w := \frac{x}{z}$)

$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} z^{\alpha_1+\alpha_2-1} \underbrace{B(\alpha_1,\alpha_2)}_{= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}} \Rightarrow$ (βλ. Εφ.)

$\Rightarrow f_{X_1+X_2}(z) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} z^{\alpha_1+\alpha_2-1} e^{-\lambda z}\ \forall z>0$, δηλ. $X_1+X_2 \sim \mathcal{G}(\alpha_1+\alpha_2, \lambda)$

και το αποτέλεσμα έπεται με επαγωγή.

β) Αν $X \sim \mathcal{G}(\alpha,\lambda)$, $c>0$ (σταθερά), τότε η τ.μ. $Y := cX \sim \mathcal{G}(\alpha, \frac{\lambda}{c})$ (βλ. Εφ. ή κάντε το — είναι άμεσο).

Άρα $2\lambda X \sim \mathcal{G}(\alpha, \frac{1}{2}) \overset{d}{=} \chi^2_{2\alpha}$, εφόσον $2\alpha \in \mathbb{N}$ (ο ορισμός της χ^2 κατανομής.)

2) $T = \frac{\sqrt{n}\, Z}{\sqrt{Y}}$, $W := \sqrt{Y/n}$ (βοηθητική) $\Big\} \Rightarrow \begin{cases} Z = TW \\ Y = nW^2 \end{cases} \Big\} \Rightarrow J = \begin{pmatrix} w & t \\ 0 & 2nw \end{pmatrix}$

$\Rightarrow \det J = 2nw^2$

$\Rightarrow f_{T,W}(t,w) = f_{Z,Y}(tw, nw^2)\,|2nw^2| \underset{Z,Y\ \text{ανεξ.}}{=} f_Z(tw) f_Y(nw^2) 2nw^2$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2w^2} \cdot \frac{(1/2)^{n/2}}{\Gamma(\frac{n}{2})} (nw^2)^{\frac{n}{2}-1} e^{-\frac{1}{2}nw^2} \cdot 2nw^2\, \mathbb{1}(w>0) =$

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3) Εφαρμόστε την (2) για $Z := \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ $(\sim N(0,1))$
και $Y := \sum_{i=1}^{n} (X_i - \bar{X})^2/\sigma^2$ $(\sim \chi^2_{n-1})$, με Z, Y ανεξ.

4) $F_{X_{nn}}(t) = P(X_{nn} \le t) = P(X_1 \le t, \dots, X_n \le t) \underset{\text{ανεξ.}}{=} \prod_{i=1}^{n} P(X_i \le t) =$
$= \prod_{i=1}^{n} F_{X_i}(t) \underset{\text{ισόνομες}}{=} F(t)^n \underset{\mathcal{U}(0,1)}{=} t^n \quad \forall t \in (0,1)$

$\Rightarrow f_{X_{nn}}(t) = F'_{X_{nn}}(t) = n t^{n-1} = \frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} t^{n-1}(1-t)^{1-1} \quad \forall t \in (0,1)$.
δηλ., $X_{nn} \sim \mathcal{B}eta(n,1)$.

Ομοίως: $1 - F_{X_{1n}}(t) = P(X_{1n} > t) = P(X_1 > t, \dots, X_n > t) =$
$\underset{\text{ανεξ.}}{=} \prod_{i=1}^{n} P(X_i > t) = \prod_{i=1}^{n} [1 - F_{X_i}(t)] \underset{\text{ισόνομες}}{=} [1 - F(t)]^n \underset{\mathcal{U}(0,1)}{=} (1-t)^n$

$\Rightarrow f_{X_{1n}}(t) = n(1-t)^{n-1} = \frac{\Gamma(1+n)}{\Gamma(1)\Gamma(n)} t^{1-1}(1-t)^{n-1} \quad \forall t \in (0,1)$.
δηλ., $X_{1n} \sim \mathcal{B}eta(1,n)$.

5) $E(Y-\theta)^2 = E(Y - EY + EY - \theta)^2 =$
$= E\{(Y-EY)^2 + (EY-\theta)^2 + 2(EY-\theta)(Y-EY)\} =$
$= E(Y-EY)^2 + E(EY-\theta)^2 + 2(EY-\theta)\underbrace{E(Y-EY)}_{=0} =$
$= \operatorname{Var} Y + (EY-\theta)^2 + 0$

6) $M_{\sum_1^n X_i}(t) = E e^{-t\sum X_i} = E \prod_1^n e^{-tX_i} \underset{\text{ανεξ.}}{=} \prod_1^n E e^{-tX_i} = \prod_1^n M_{X_i}(t)$ (*)

(*) $\Rightarrow M_{\sum_1^n X_i}(t) = \prod_{i=1}^{n} e^{\theta_i(e^{-t}-1)} = \exp\{(\sum_1^n \theta_i)(e^{-t}-1)\}$
$\Rightarrow \sum X_i \sim \mathcal{P}(\sum_1^n \theta_i)$

Για την αποδ. χωρίς χρήση ροπογεννητριών βλ. Σημ. Πιθανοτήτων Παράδ. 6.25^{ν} / σ. 118. Μετά επαγωγή.

7) Από την (*) της 6, έχουμε:

$$M_{Y_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \left(pe^{-t}+(1-p)\right)^{m_i} = \left(pe^{-t}+(1-p)\right)^{\sum_1^n m_i}$$

$$\Rightarrow Y_n \sim Bi\left(\sum_1^n m_i, p\right)$$

8) Από (*) της 6, έχουμε:

$$M_{Y_n}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \left[\frac{pe^{-t}}{1-(1-p)e^{-t}}\right]^{k_i} =$$

$$= \left[\frac{pe^{-t}}{1-(1-p)e^{-t}}\right]^{\sum_1^n k_i} \Rightarrow Y_n \sim NBi\left(\sum_1^n k_i, p\right).$$