Πιθανότητες ΙΙ

Λύσεις 10

1) $$P(X>Y)=\int_0^{+\infty}\int_y^{+\infty}\lambda e^{-\lambda x}\frac{\lambda^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\lambda y}\,dx\,dy$$

$$=\int_0^{+\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)}\,y^{\alpha-1}e^{-\lambda y}\left(\int_y^{+\infty}\lambda e^{-\lambda x}dx\right)dy$$

$$=\int_0^{+\infty}\frac{\lambda^\alpha}{\Gamma(\alpha)}\,y^{\alpha-1}e^{-\lambda y}e^{-\lambda y}dy$$

$$=\frac{1}{2^\alpha}\int_0^{+\infty}\frac{(2\lambda)^\alpha}{\Gamma(\alpha)}\,y^{\alpha-1}e^{-2\lambda y}dy=\frac{1}{2^\alpha}$$

2) $$P(2X<Y)=\int_0^{+\infty}\int_{2x}^{+\infty}2e^{-(x+y)}\,dy\,dx$$

$$=2\int_0^{+\infty}e^{-x}\left(\int_{2x}^{+\infty}e^{-y}dy\right)dx=2\int_0^{+\infty}e^{-x}\cdot e^{-2x}dx$$

$$=\frac{2}{3}\int_0^{+\infty}3e^{-3x}dx=\frac{2}{3}$$

3) $P(X < Z < Y)$

$$= \int_0^{+\infty} \int_x^{+\infty} \int_x^{y} e^{-z}\, 2e^{-(x+y)} \, dz \, dy \, dx$$

$$= \int_0^{+\infty} \int_x^{+\infty} 2e^{-(x+y)} \left(\int_x^{y} e^{-z} dz \right) dy \, dx$$

$$= \int_0^{+\infty} \int_x^{+\infty} 2e^{-(x+y)} \left(e^{-x} - e^{-y} \right) dy \, dx$$

$$= \int_0^{+\infty} \int_x^{+\infty} \left(2e^{-2x-y} - 2e^{-x} e^{-2y} \right) dy \, dx$$

$$= \int_0^{+\infty} \left(2e^{-2x} \int_x^{+\infty} e^{-y} dy - e^{-x} \int_x^{+\infty} 2e^{-2y} dy \right) dx$$

$$= \int_0^{+\infty} \left(2e^{-2x} e^{-x} - e^{-x} e^{-2x} \right) dx$$

$$= \int_0^{+\infty} e^{-3x} dx = \frac{1}{3}$$

4) $P\left(\frac{Y}{X} \le Z\right)$

$$= \int_0^{+\infty} \left(\int_0^{+\infty} \left(\int_{\frac{y}{z}}^{+\infty} e^{-x} dx \right) y e^{-yz} dz \right) e^{-y} dy$$

$$= \int_0^{+\infty} \int_0^{+\infty} e^{-\frac{y}{z}} \, y e^{-yz} e^{-y} \, dz \, dy$$

$$= \int_0^{+\infty} \int_0^{+\infty} y \, e^{-y\left(z+\frac{1}{z}+1\right)} \, dy \quad dz$$

$$= \int_0^{+\infty} \frac{1}{\left(z+\frac{1}{z}+1\right)^2} dz = \int_0^{+\infty} \frac{z^2}{(z^2+z+1)^2} dz = I$$

$$I = \int_0^{+\infty} \frac{dz}{z^2+z+1} - \int_0^{+\infty} \frac{z+1}{(z^2+z+1)^2} dz$$

$$= \int_0^{+\infty} \frac{dz}{z^2+z+1} - \frac{1}{2} \int_0^{+\infty} \frac{2z+1}{(z^2+z+1)^2} dz - \frac{1}{2} \underbrace{\int_0^{+\infty} \frac{dz}{(z^2+z+1)^2}}$$

$z = \frac{1}{t}$

$$\int_0^{+\infty} \frac{dz}{(z^2+z+1)^2} = \int_{+\infty}^{0} -\frac{1}{t^2} \frac{dt}{\left(\frac{1}{t^2}+\frac{1}{t}+1\right)^2} = \int_0^{+\infty} \frac{t^2 dt}{(t^2+t+1)^2} = I$$

$$\frac{3}{2} I = \int_0^{+\infty} \frac{dz}{z^2+z+1} - \frac{1}{2} \int_0^{+\infty} \frac{2z+1}{(z^2+z+1)^2} dz$$

$$\int_0^{+\infty} \frac{dz}{z^2+z+1} = \int_0^{+\infty} \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{4}{3} \int_0^{+\infty} \frac{dz}{\left(\frac{2}{\sqrt{3}} z+\frac{1}{\sqrt{3}}\right)^2+1}$$

$$= \frac{2}{\sqrt{3}} \int_0^{+\infty} \frac{d\left(\frac{2}{\sqrt{3}} z+\frac{1}{\sqrt{3}}\right)}{1+\left(\frac{2}{\sqrt{3}} z+\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}} z+\frac{1}{\sqrt{3}}\right)\Big|_0^{+\infty} = \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}$$

$$\int_0^{+\infty} \frac{2z+1}{\left(z^2+z+1\right)^2} dz = -\frac{1}{z^2+z+1}\Big|_0^{+\infty} = 1$$

$$I = \frac{2}{3}\left(\frac{2\pi}{3\sqrt{3}} - \frac{1}{2}\right) = \frac{4\pi}{9\sqrt{3}} - \frac{1}{3}$$

5) $$P(X_{1N} \geq x) = \sum_{k=0}^{\infty} P(N=k)\, P(X_{1N} > x \mid N=k)$$

$(x>0)$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \left(P(X_1 > x)\right)^k$$

$$= \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} e^{-kx} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{(\lambda e^{-x})^k}{k!}$$

$$= e^{-\lambda} \left(e^{\lambda e^{-x}} - 1\right)$$

$$F_{X_{1N}}(x) = \left(1 - e^{-\lambda(1-e^{-x})} + e^{-\lambda}\right) 1_{(x>0)}$$

$$f_{X_{1N}}(x) = e^{-\lambda(1-e^{-x})} \lambda e^{-x} 1_{(x>0)}$$

6) (a) $$EY = \sum_{k=1}^{\infty} P(N=k)\ E\left(\sum_{i=1} X_i \mid N=k\right)$$

$$= \sum_{k=1}^{\infty} p(1-p)^{k-1}\, k = \frac{1}{p}$$

$$\operatorname{Var} Y = \sum_{k=1}^{\infty} P(N=k)\ \operatorname{Var}\left(\sum_{i=1}^{N} X_i \,\Big|\, N=k\right)$$

$$= \sum_{k=1}^{\infty} p(1-p)^{k-1}\, k\ \frac{1-p}{p^2}$$

$$= \frac{1-p}{p^3}$$

$$\operatorname{Cov}(Y, N) = \sum_{k=1}^{\infty} P(N=k)\ \operatorname{Cov}(Y, N \mid N=k)$$

$$= \sum_{k=1}^{\infty} P(N=k)\ \operatorname{Cov}(X_1 + \cdots + X_k,\ N) = 0$$

(b) $$\underset{(y>0)}{P(Y \le y)} = \sum_{k=1}^{\infty} P(N=k)\ P\left(X_1 + \cdots + X_k \le y\right)$$

$$= \sum_{k=1}^{\infty} P(N=k)\ P\left(\mathcal{G}(k,1) \le y\right)$$

$$\underset{(y>0)}{f_Y(y)} = \sum_{k=1}^{\infty} p(1-p)^{k-1}\, \frac{\lambda^k}{\Gamma(k)}\, y^{k-1} e^{-\lambda y}$$

$$= p\lambda e^{-\lambda y} \sum_{k=0}^{\infty} \frac{\left[(1-p)\lambda y\right]^k}{k!}$$

$$= p\lambda e^{-\lambda y}\, e^{(1-p)\lambda y} = p\lambda e^{-p\lambda y} = \mathcal{E}xp(p\lambda)$$