Πιθανότητες ΙΙ

Λύσεις 9

1) (α) Για $z>0$:

$$f_Z(z) = \int_0^{+\infty} f_{Z|Y}(z|y)\, f_Y(y)\,dy$$

$$= \int_0^{+\infty} \frac{y^n}{\Gamma(n)} z^{n-1} e^{-yz} \lambda e^{-\lambda y}\,dy$$

$$= \frac{z^{n-1}\lambda}{\Gamma(n)} \frac{\Gamma(n+1)}{(\lambda+z)^{n+1}} \int_0^{+\infty} \frac{(\lambda+z)^{n+1}}{\Gamma(n+1)} y^n e^{-(\lambda+z)y}\,dy$$

$$= \frac{\lambda z^{n-1} n}{(\lambda+z)^{n+1}}$$

$$f_{Y|Z}(y|z) = \frac{f_2(y,z)}{f_Z(z)} = \frac{f_Y(y)\, f_{Z|Y}(z|y)}{f_Z(z)}$$

$(y>0,\ z>0)$

$$= \frac{\lambda e^{-\lambda y}\, y^n\, z^{n-1}\, e^{-yz}\, (\lambda+z)^{n+1}}{\Gamma(n)\, \lambda z^{n-1} n}$$

$$= \frac{(\lambda+z)^{n+1}}{\Gamma(n+1)} y^n e^{-(\lambda+z)y}$$

(β) $E(Y|Z=z) = E\left(G(n+1, \lambda+z)\right) = \dfrac{n+1}{\lambda+z}$

2) (a) $P(Z=z) = \sum_{y=1} P(Y=y)\ P(Z=z \mid Y=y)$

$(z \in \mathbb{N}_0)$

$$= \sum_{y=1}^{\infty} p\,(1-p)^{y-1}\ e^{-y}\ \frac{y^z}{z!}$$

$$= \frac{p}{1-p}\ \frac{1}{z!} \sum_{y=1}^{\infty} \left(\frac{1-p}{e}\right)^{y} y^{z}$$

$$P(Y=y \mid Z=z) = \frac{P(Y=y)\ P(Z=z \mid Y=y)}{P(Z=z)}$$

$$= \frac{p\,(1-p)^{y-1}\ e^{-y}\ \frac{z^y}{y!}}{\frac{p}{1-p}\ \frac{1}{z!} \sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}}$$

$$= \frac{(1-p)^{y} \left(\frac{z}{e}\right)^{y} z!}{y! \sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}}\ ,\ y \in \mathbb{N},\ z \in \mathbb{N}_0$$

(b) $E(Y \mid Z=z)$

$$= \sum_{y=1}^{\infty} y\ P(Y=y \mid Z=z)$$

$$= \sum_{y=1}^{\infty} \frac{(1-p)^{y} \left(\frac{z}{e}\right)^{y} z!}{(y-1)! \left[\sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}\right]}$$

$$= \frac{z!}{\sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}} \cdot \sum_{y=1}^{\infty} \frac{\left(\frac{(1-p)z}{e}\right)^{y}}{(y-1)!}$$

$$= \frac{z!}{\sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}} \sum_{y=0}^{\infty} \frac{\left[\frac{(1-p)z}{e}\right]^{y+1}}{y!}$$

$$= \frac{z! \, \frac{(1-p)z}{e}}{\sum_{k=1}^{\infty} \left(\frac{1-p}{e}\right)^{k} k^{z}} \; e^{\frac{(1-p)z}{e}}$$