DIVISORS ON HURWITZ SPACES: AN APPENDIX TO 'THE CYCLE CLASSES OF DIVISORIAL MARONI LOCI'

GERARD VAN DER GEER AND ALEXIS KOUVIDAKIS

Abstract. The Maroni stratification on the Hurwitz space of degree d covers of genus g has a stratum that is a divisor only if $d-1$ divides g. Here we construct a stratification on the Hurwitz space that is analogous to the Maroni stratification, but has a divisor for all pairs (d,g) with $d \leq g$ with a few exceptions and we calculate the divisor class of an extension of these divisors to the compactified Hurwitz space.

1. Introduction

The Hurwitz space $\mathcal{H}_{d,g}$ of simply-branched covers of genus g and degree d carries a stratification named after Maroni ([7]) that is defined as follows. If $\gamma : C \to \mathbb{P}^1$ is a simply-branched cover one takes the dual of the cokernel of the natural map

$$\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C$$

which is a vector bundle of rank $d-1$ on the projective line, hence is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$ for some $(d-1)$-tuple $\alpha = (a_1, \ldots, a_{d-1})$, where we assume that the a_i are non-decreasing. The loci of covers $\gamma : C \to \mathbb{P}^1$ with fixed α are the strata.

It is known (see [1]) that for general $\gamma : C \to \mathbb{P}^1$ of genus

$$g = k(d-1) + s \qquad \text{with } 0 \leq s \leq d-2$$

the tuple α takes the form $(k+1, \ldots, k+1, k+2, \ldots, k+2)$ with s enties equal to $k+2$. Only the case with $s = 0$ yields a Maroni stratum that is a divisor (see [3] and [9, Thm. 1.15]).

In this paper we show how to define for the case that the genus g is not divisible by $d-1$ a stratification that has a stratum that is a divisor for $g \geq d$ under exclusion of a few cases. If $d-1$ divides g then this reduces to the stratification of Maroni loci. It uses instead of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C$ the cokernel of a natural map

$$\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C(D)\,,$$

where D is an appropriately chosen divisor of degree s with support in the ramification locus of γ. The cycle classes of an extension of these divisors to the compactified Hurwitz space $\overline{\mathcal{H}}_{d,g}$ can be calculated by using a global-to-local evaluation map $p^* p_* V \to V$ of a vector bundle V on an extension of the $\mathbb{P}^1$-fibration $p : \mathbb{P} \to \mathcal{H}_{d,g}$ to the compactified Hurwitz space that is trivial on the generic fibre of p. The calculation and the answer are completely analogous to case of the cycle classes of the Maroni divisors calculated in [4]. The cycle classes are given in terms of an explicit sum of boundary divisors.

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2. The Setting

We recall the setting from [4]. We denote by $\overline{\mathcal{H}}_{d,g}$ the compactified Hurwitz space of admissible covers of degree d and genus g. We have

$$\overline{\mathcal{H}}_{d,g} - \mathcal{H}_{d,g} = \cup_{j,\mu} S_{j,\mu},$$

where the $S_{j,\mu} = S_{b-j,\mu}$ are divisors indexed by $2 \leq j \leq b-2$ and a partition $\mu = (m_1, \ldots, m_n)$ of d. These divisors can be reducible, but a generic point corresponds to an admissible cover $\gamma : C \to P$ where P is a genus 0 curve consisting of two components P_1, P_2 of genus 0 intersecting in one point Q with $j_1 = j$ or $b-j$ branch points on P_1 (resp. $j_2 = b-j$ or j branch points on P_2) and the inverse image $\gamma^{-1}(Q)$ consists of n points $Q_1, \ldots, Q_n$ on C with ramification indices $m_1, \ldots, m_n$. Since $\overline{\mathcal{H}}_{d,g}$ is not normal we normalize it and this results in a smooth stack $\widetilde{\mathcal{H}}_{d,g}$. We then have a diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "c"] \arrow[d, "\varpi"'] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

where $\widetilde{\mathcal{C}}$ is the universal curve and $\overline{\mathcal{M}}_{0,b}$ is the moduli space of stable b-pointed curves of genus 0 and π_{b+1} is the map that forgets the $(b+1)$st point. With $\mathbb{P}$ the fibre product of $\overline{\mathcal{M}}_{0,b+1}$ and $\widetilde{\mathcal{H}}_{d,g}$ over $\overline{\mathcal{M}}_{0,b}$ we have a diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\alpha"] \arrow[dr, "\varpi"'] & \mathbb{P} \arrow[r, "c'"] \arrow[d, "\varpi'"] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ & \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b} \end{tikzcd}$$

We now work over a base B (it can be $\mathcal{H}_{d,g}$, $\widetilde{\mathcal{H}}_{d,g}$ or often a 1-dimensional base). But we shall suppress the index B. Note that normalization commutes with base change. In [4, Lemma 4.1] we showed that $\widetilde{\mathcal{C}}$ and $\mathbb{P}$ have only singularities of type A_k. We resolve the singularities of $\mathbb{P}$ obtaining a space $\widetilde{\mathbb{P}}$ and then let Y be the normalization of $\widetilde{\mathcal{C}} \times_{\mathbb{P}} \widetilde{\mathbb{P}}$ and let $\widetilde{Y}$ be the resolution of singularities of Y. We then find the basic diagram as in [4]:

Diagram 2.1.

$$\begin{tikzcd}
\widetilde{Y} \arrow[dr, "\nu"] \arrow[drr, bend left, "\rho"] \arrow[ddr, "\tilde{\pi}"'] & & & \\
& Y \arrow[r] \arrow[d, "\pi"] & \widetilde{\mathcal{C}} \arrow[d] \arrow[ddr, "\varpi"] & \\
& \widetilde{\mathbb{P}} \arrow[r] \arrow[drr, "p"'] & \mathbb{P} \arrow[dr] & \\
& & & B
\end{tikzcd}$$

We observe that the finite map $\pi : Y \to \widetilde{\mathbb{P}}$ is flat as Y is Cohen-Macaulay and $\widetilde{\mathbb{P}}$ is smooth. Actually, Y has rational singularities only.

3. Constructing Divisors

Let $\mathcal{D}$ be an effective divisor on Y of relative degree s over B, supported on the sections. This is a Cartier divisor since the sections do not intersect the singular locus and so $\mathcal{O}(\mathcal{D})$ is a line bundle on Y. Therefore it follows that $\pi_*\mathcal{O}(\mathcal{D})$ is a locally free sheaf on $\widetilde{\mathbb{P}}$. We denote by $\tilde{\mathcal{D}}$ the proper transform of $\mathcal{D}$ under the resolution map ν. Since $\nu_*\mathcal{O}(\tilde{\mathcal{D}}) = \nu_*\mathcal{O}_{\tilde{Y}} \otimes \mathcal{O}(\mathcal{D})$, we conclude by [4, Lemma 4.4] and the fact that $R^j\nu_*\mathcal{O}_{\widetilde{Y}} = 0$ for $j \geq 1$, that $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}) = \pi_*\mathcal{O}(\mathcal{D})$. We can use the restriction of $\pi_*(\mathcal{O}(\mathcal{D}))$ to the open part over $\mathcal{H}_{d,g}$ to define a stratification by type of the bundle on $\mathbb{P}^1$ just as for the Maroni stratification. We are interested in the case we get a divisor.

For a divisor $\mathcal{D}$ we have an inclusion

$$\iota_{\mathcal{D}} : \mathcal{O}_{\tilde{\mathbb{P}}} \to \tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})\,.$$

Note that the image $\iota_{\mathcal{D}}(1)$ of the section 1 is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$.

We now introduce the vector bundle that we use to define a stratification.

Definition 3.1. We let $\mathcal{K}_{\mathcal{D}}$ be the cokernel of $\iota_{\mathcal{D}}$. We define $V_{\mathcal{D}} := \mathcal{K}_{\mathcal{D}}^{\vee}$ as the dual $\mathcal{O}_{\tilde{\mathbb{P}}}$-module. Since $\iota_{\mathcal{D}}(1)$ is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$ the sheaf $\mathcal{K}_{\mathcal{D}}$ is locally free of rank $d-1$ on $\tilde{\mathbb{P}}$ and therefore so is $V_{\mathcal{D}}$.

We start with a lemma which follows immediately from the Riemann-Roch theorem.

Lemma 3.2. *Let $U = \oplus_{i=1}^{r}\mathcal{O}(a_i)$ be a vector bundle of rank r and degree n on $\mathbb{P}^1$. Suppose that $-1 \leq a_1 \leq \cdots \leq a_r$. Then $h^0(V) = r + n$. Moreover, this is the minimum dimension for the space of sections of a vector bundle of rank r and degree n on $\mathbb{P}^1$.*

Given a cover $\gamma : C \to \mathbb{P}^1$ and an effective divisor D of degree s supported on the ramification divisor of γ, we write

$$\gamma_*\mathcal{O}_C(D) \cong \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{d-1}),$$

with $a_0 \leq a_1 \leq \cdots \leq a_{d-1}$. Note that $\sum_{i=1}^{d-1} a_i = (k+1)(d-1)$.

We let $\gamma : C \to \mathbb{P}^1$ represent a general point of $\mathcal{H}_{d,g}$. We want to determine the numbers a_i. We know by results of Ballico [1] that

$$\gamma_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+1))^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+2))^{\oplus s}.$$

Thus we get

$$\gamma_* \mathcal{O} \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus s}.$$

so that $h^0(k\, g^1_d) = k+1$ and $h^1(k\, g^1_d) = s$.

Proposition 3.3. *For $\gamma : C \to \mathbb{P}^1$ a general point of $\mathcal{H}_{d,g}$ we can choose an effective divisor D of degree s supported on the ramification locus R of γ satisfying*

$$h^0(D) = 1 \quad \textit{and} \quad h^0(k\, g^1_d + D) = k+1.$$

Proof. Note that since for a general γ we have $h^0(k\, g^1_d) = k+1$, the first condition is implied by the second. This is because for any effective divisors D_1, D_2 on C we have that $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$. Observe that $b - (2g-2) = 2\,d$. We consider the linear system $|K_C - k\, g^1_d|$. For reasons of degree we can choose a ramification point p_1 which is not a base point of $|K_C - k\, g^1_d|$ and this gives $h^0(K_C - k\, g^1_d - p_1) = s - 1$. Then by the same degree argument we can find a ramification point p_2 such that it is not a base point of $|K_C - k\, g^1_d - p_1|$ such that $h^0(K_C - k\, g^1_d - p_1 - p_2) = s - 2$. Repeating the argument we arrive at a divisor of degree s supported on the ramification locus such that $h^0(K_C - k\, g^1_d - D) = 0$, hence by duality $h^1(k\, g^1_d + D) = 0$. By Riemann-Roch we have $h^0(k\, g^1_d + D) = k+1$. $\square$

Now if we choose D as in Proposition 3.3 we have $h^0(D) = 1$ and therefore $a_0 = 0$ and $a_i > 0$ for $i \geq 1$. Moreover $h^0(k\, g^1_d + D) = k+1$ and since

$$\gamma_* \mathcal{O}_C(D) \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_{d-1})$$

this implies that $a_i \geq k+1$ for all $i = 1, \cdots, d-1$. Since the a_i add up to $(k+1)(d-1)$ we conclude that all $a_i = k+1$.

Conclusion 3.4. *If we choose γ and D as in Proposition 3.3 then the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C(D)$ has type $\mathcal{O}(k+1)^{\oplus d-1}$.*

We now want to see that our degeneracy locus is non-empty in the open Hurwitz space $\mathcal{H}_{d,g}$. According to Ohbuchi [8] only so-called acceptable $(d-1)$-tuples $(a_1, \cdots, a_{d-1})$ can occur as the indices of the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C$; here acceptable is defined as follows, see [9].

Definition 3.5. A non-decreasing $(d-1)$-tuple of natural numbers $(a_1, \cdots, a_{d-1})$ with $\sum_{i=1}^{d-1} a_i = b/2$ is said to be acceptable for (d, g) if the a_i satisfy

(1) $a_1 \geq b/d(d-1)$;
(2) $a_{d-1} \leq b/d$;
(3) $a_{i+1} - a_1 \leq a_1$.

We now consider the unique acceptable $(d-1)$-tuple α for which with $a_1 = k$ and for which the sum

$$\sum_{i=1}^{d-1} (d-i) a_i$$

is maximal. This means that this α is the most balanced $(d-1)$-tuple with $a_1 = k$. In [2] Coppens proved the following existence result, see also [9].

Theorem 3.6. *Let a_1 be an integer satisfying (2) of Definition 3.5. If α is the unique acceptable $(d-1)$-tuple $(a_1, \cdots, a_{d-1})$ for which the sum $\sum_{i=1}^{d-1}(d-i)a_i$ is maximal, then the locus in $\mathcal{H}_{d,g}$ of covers $\gamma : C \to \mathbb{P}^1$ with invariant α is not empty.*

We apply this in the case $a_1 = k$ and $\sum_{i=1}^{d-1}(d-i)a_i = (k+1)(d-1)+s$. We find the following maximizing $(d-1)$-tuples for the a_i of $V = \oplus_{i=1}^{d-1}\mathcal{O}(a_i)$ with $a_1 = k$.

Lemma 3.7. *If $a_1 = k$ we have the following maximizing acceptable $(d-1)$-tuples:*
1. *if $s \le d-4$ the maximizing sequence is $(k, (k+1)^{d-s-3}, (k+2)^{s+1})$;*
2. *if $s = d-3$ the maximizing sequence is $(k, (k+2)^{d-2})$, with $g \ne 2(d-2)$;*
3. *if $s = d-2$ the maximizing sequence is $(k, (k+2)^{d-3}, k+3)$, with $g \ne 2d-3$ and $(d, g) \ne (3, 5)$.*

We consider the three cases of Lemma 3.7.

Lemma 3.8. *Let $\gamma : C \to \mathbb{P}^1$ be a general cover of type (1), (2) or (3) of Lemma 3.7. Then there exists a divisor of degree s with support in the ramification locus of γ such that $h^0(k\, g^1_d) = h^0(k\, g^1_d + D)$ and $h^0(D) = 1$.*

Proof. In the first case we consider a general cover $\gamma : C \to \mathbb{P}^1$ of this type. Then we have by [1]

$$\gamma_*\mathcal{O}_C = \mathcal{O} \oplus \mathcal{O}(-k) \oplus \mathcal{O}(-(k+1))^{\oplus d-s-3} \oplus \mathcal{O}(-(k+2))^{\oplus s+1}$$

so that $h^0(k\, g^1_d) = k+2$ and accordingly $h^1(k\, g^1_d) = s+1$. We follow the argument of Proposition 3.3. The argument for the cases (2) and (3) is similar. □

For this pair (γ, D) the conditions $h^0(D) = 1$, $h^0(kg^1_d + D) = k+2$ imply that for

$$\gamma_*\mathcal{O}(D) = \oplus_{i=0}^{d-1}\mathcal{O}(-a_i)$$

we must have $a_0 = 0$, $a_1 = k$ and $a_i \ge k+1$ for $i = 2, \cdots, d-1$. Then the twisted (with $\mathcal{O}(-k-1)$) dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C(D)$ has the form

$$\oplus_{i=1}^{d-1}\mathcal{O}(b_i) \ \text{ with } \ b_1 = -1 \ \text{ and } \ b_i \ge 0 \ \text{ for } \ i = 2, \cdots, d-1$$

and thus is not balanced with a space of sections of minimum dimension (see Lemma 3.2). The other two cases are dealt with similarly.

Conclusion 3.9. *Let $\gamma : C \to \mathbb{P}^1$ be a general cover such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C$ has type as in Lemma 3.7. Then there exists a divisor D of degree s with support in the ramification divisor of γ such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C(D)$ has unbalanced type as above.*

We define now our divisors that are analogous to the Maroni divisors. We consider the pullback of the diagram 2.1 to the open base $B = \mathcal{H}_{d,g}$ and choose on Y_B a reduced divisor $\mathcal{D}$ consisting of any s sections of $\tilde{\pi}|_B$ and we let $V_{\mathcal{D}}$ be the dual of the cokernel of the natural map $\iota_{\mathcal{D}}|_B : \mathcal{O}_{\tilde{Y}_B} \to \tilde{\pi}_*\mathcal{O}(\mathcal{D})$ and tensor $V_{\mathcal{D}}$ with a line bundle M on $\tilde{\mathbb{P}}_B$ such that $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$ has zero degree on the generic fibre of π. The sheaf $p_*V'_{\mathcal{D}}$ is reflexive (see [5]) and therefore it is a vector bundle on an open U of $\mathcal{H}_{d,g}$ with complement of codimension ≥ 3.

Consider an open subset U' of U such that $p_*V'_{\mathcal{D}}$ is a trivial of rank r, that is, isomorphic to $\mathcal{O}^r_{U'}$. Choose r generating sections $s_1, \dots, s_r$ of $p_*V'_{\mathcal{D}}$ on U'. If we consider their pull backs under p and restrict these for $x \in U'$ to $H^0(p^{-1}(x), V_{\mathcal{D}}|p^{-1}(x))$ then they generate the stalk of $V'_{\mathcal{D}}$ at a point of $p^{-1}(x)$ if and only if $V'_{\mathcal{D}}|p^{-1}(x)$

is the trivial bundle of rank r on $p^{-1}(x) = \mathbb{P}^1$; indeed, $V'_{\mathcal{D}}|p^{-1}(x)) = \oplus_i \mathcal{O}(a_i)$ with $\sum_i a_i = 0$ and if some a_i are negative, then these sections cannot generate it since these do not see the $\mathcal{O}(a_i)$ with a_i negative; if $V'_{\mathcal{D}}|p^{-1}(x) = \mathcal{O}^r_{\mathbb{P}^1}$ by Grauert's theorem they will generate it (see e.g. [6, III,Cor. 12.9]). Thus we arrive at the following result.

Theorem 3.10. *Suppose that $g = k(d-1) + s$ with $0 \leq s \leq d-2$ and $3 \leq d \leq g$, satisfying the conditions that $g \neq 2d-3$, $g \neq 2d-4$ and $(d, g) \neq (3, 5)$. Then the vanishing locus of the determinant of the evaluation map $\mathrm{ev} : p^* p_* V'_{\mathcal{D}} \to V'_{\mathcal{D}}$ defines a non-empty divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$.*

Proof. We take a cover as in Conclusion 3.4. Then there is a point in $\mathcal{H}_{d,g}$ representing this cover and such that the divisor D of Conclusion 3.4 is given by the restriction of the above $\mathcal{D}$ to the corresponding fiber. This is because the points of $\mathcal{H}_{d,g}$ parametrize simply branched coverings with ordered branch points, and hence the ramification points are ordered too. We conclude that $V'_{\mathcal{D}}$ is trivial on the generic fibre. But by Conclusion 3.9 above we see, for similar reasons, that the locus in $\mathcal{H}_{d,g}$, where $V'_{\mathcal{D}}$ restricted to a fibre is non-trivial is not empty. By Lemma 3.2 and Grauert's theorem both the above loci belong to U. Thus the evaluation map on $p^{-1}U$ is a map of vector bundles of the same rank with degeneracy locus which is not the whole space nor empty. Therefore it defines a divisor on $p^{-1}U$ which extends uniquely to a divisor on $\mathcal{H}_{d,g}$. □

4. Cycle Classes

In this section we indicate how to calculate the classes of closed divisors in $\overline{\mathcal{H}}_{d,g}$ that extend the above defined divisors $\mathfrak{d}_{\mathcal{D}}$. We proceed as in [4] for the Maroni case.

We take the standard line bundle M constructed in [4, Definition 6.2] and we let $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$. We then want to find a divisor $A_{\mathcal{D}}$ supported on the singular fibres of $p : \tilde{Y} \to B$ such that $c_1(V'_{\mathcal{D}}) - A_{\mathcal{D}}$ is a pullback under p from the base B. In [4, Section 6] we constructed such a divisor $A_{\mathcal{D}}$ in the case where the divisor $\mathcal{D}$ is zero. This is done locally on the base B. We restrict to 1-dimensional bases B and consider the fibre of p over a points s where B intersects the boundary. The fibre there consists of a chain of rational curves. In the present case the degrees of V', the bundle used in [4] and corresponding to $\mathcal{D} = 0$, and $V'_{\mathcal{D}}$ differ at a chain $R_0 = P_1, R_1, \ldots, R_m = P_2$ only at R_0 and R_m. Adapting the result gives the following analogue of [4, Conclusion 6.5].

Proposition 4.1. *For each irreducible component Σ of $S_{j,\mu}$ we define*

$$c_{\Sigma,\mathcal{D}} = d - n - 2(r - \mathcal{D} \cdot P_2).$$

If we let

$$A^{\Sigma}_{\mathcal{D}} = -\frac{1}{2} \sum_{i=0}^{m-1} ((m-i)\, c_{\Sigma,\mathcal{D}} - \delta_i) R^{\Sigma}_i$$

then the degree of $A^{\Sigma}_{\mathcal{D}}$ on R^{Σ}_i equals the degree of $V'_{\mathcal{D}}$ on R^{Σ}_i.

The formalism described in [4] shows that if Q denotes the degeneracy locus of the evaluation map $\mathrm{ev} : p^* p_* V'_{\mathcal{D}} \to V'_{\mathcal{D}}$ we get that

$$c_1(Q) + p^* R^1 p_* V'_{\mathcal{D}} + (-A_{\mathcal{D}})_{\mathrm{sh}},$$

where the index sh denotes the shift as in [4, Definition 3.7], is an effective class on $\widetilde{\mathbb{P}}$ that is a pull back under p, and pulling it back under a section of p gives us an effective class $\overline{\mathfrak{d}}_{\mathcal{D}}$ which extends the divisor $\mathfrak{d}_{\mathcal{D}}$.

To calculate the class $\overline{\mathfrak{d}}_{\mathcal{D}}$ of the degeneracy locus we wish to use the formula of [4, Theorem 3.10]. For this we need first a the following lemma (see [4, Proposition 10.1]).

Lemma 4.2. *Let $\mathcal{L}$ be a line bundle on $\tilde{Y}$ with first Chern class ℓ. If U denotes the ramification divisor of $\tilde{\pi}$ we have*

$$c_1(\tilde{\pi}_*\mathcal{L}) = \tilde{\pi}_*\ell - \frac{1}{2}\tilde{\pi}_*(U),$$
$$c_2(\tilde{\pi}_*\mathcal{L}) - c_2(\pi_*\mathcal{O}_{\tilde{Y}}) = \frac{1}{2}((\tilde{\pi}_*\ell)^2 - \tilde{\pi}_*(\ell^2)) - \frac{1}{2}(\tilde{\pi}_*U \cdot \tilde{\pi}_*\ell - \tilde{\pi}_*(U \cdot \ell)).$$

We denote the sections of p by σ_i $(i = 1, \ldots, b)$ and their images by Ξ_i $(i = 1, \ldots, b)$. These are divisors on $\tilde{\mathbb{P}}$. Over the section Ξ_i we have a ramification divisor U_i. If $\mathcal{D}_1$ is a reduced divisor with support on the sections $\cup_i \Xi_i$ we let $\mathcal{D}$ be the corresponding divisor with support on the ramification divisor of $\tilde{\pi}$. Then we have

$$\tilde{\pi}_*\mathcal{D} = \mathcal{D}_1, \quad \tilde{\pi}^*\mathcal{D}_1 = 2\,\mathcal{D} + \Gamma_{\mathcal{D}},$$

with $\Gamma_{\mathcal{D}}$ disjoint from $\mathcal{D}$. Another relation that we will use with $W = \tilde{\pi}_*(U)$ is

$$U \cdot \mathcal{D} = \frac{1}{2}W \cdot \mathcal{D}_1 = \frac{1}{2}\mathcal{D}_1^2.$$

Using Lemma 4.2 we thus find

$$c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}), \quad c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}).$$

Therefore, when we apply [4, Theorem 3.10] in order to calculate the class $\overline{\mathfrak{d}}_{\mathcal{D}}$ the only thing that differs from the case treated there is the choice of the line bundle $A_{\mathcal{D}}$. This affects only the definition of $c_{\Sigma,\mathcal{D}}$ and therefore the class is calculated by the same formula given in [4, Theorem 8.3] with $c_{j,\mu}$ determined by $c_{\Sigma,\mathcal{D}}$.

We can twist the bundle $V'_{\mathcal{D}}$ by a line bundle N corresponding to a divisor on $\widetilde{\mathbb{P}}$ with support on the boundary of $\widetilde{\mathbb{P}}$. This results in a divisor class $\overline{\mathfrak{d}}_{\mathcal{D},N}$ on $\overline{\mathcal{H}}_{d,g}$ extending the divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$. We thus arrive as in [4, Section 9] at the following theorem.

Theorem 4.3. *Suppose that $3 \leq d \leq g$ and $g \neq 2d-3$, $g \neq 2d-4$ and $(d,g) \neq (3,5)$. Let Σ be an irreducible component of the boundary $S_{j,\mu}$ of $\overline{\mathcal{H}}_{d,g}$. Then the coefficient $\sigma_{j,\mu}$ of Σ of the locus $\cap_N \overline{\mathfrak{d}}_{\mathcal{D},N}$ is equal to*

$$m(\mu)\left(\frac{1}{12}(d - \sum_{\nu=1}^{n(\mu)}\frac{1}{m_\nu}) + \frac{j(b-j)(d-2)}{8(b-1)(d-1)}\right) - \frac{1}{8(d-1)}\sum_{i=1}^{m(\mu)}(\delta_{i-1}-\delta_i)^2$$
$$-\frac{d-1}{2}\left(\frac{m(\mu)}{4} - \sum_{i=1}^{m(\mu)}(e_{i-1}-e_i)^2\right).$$

For the numbers $m(\mu)$, δ_i and e_i we refer to [4]. Note that the rational numbers e_i that come from rounding off a rational solution to an integral one depend on the constants $c_{j,\mu}$ that occur above.

In a similar way one can obtain analogues of the theorems of Sections 10 and 11 of [4] by adding to $\mathcal{D}$ an effective divisor Z supported on the boundary of $\widetilde{Y}$. One can then define an effective class $\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$ that extends the class of $\mathfrak{d}_{\mathcal{D}}$. One may ask whether

$$\cap_{N,Z}\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$$

is the class of the Zariski closure of $\mathfrak{d}_{\mathcal{D}}$ on $\overline{\mathcal{H}}_{d,g}$.

The classes $\mathfrak{d}_{\mathcal{D}}$ depend on $\mathcal{D}$, but using the monodromy one sees that their images on the Hurwitz space with unordered branch points do not.

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Korteweg-de Vries Instituut, Universiteit van Amsterdam, Postbus 94248, 1090 GE Amsterdam, The Netherlands

E-mail address: `G.B.M.vanderGeer@uva.nl`

Department of Mathematics and Applied Mathematics, University of Crete, GR-70013 Heraklion, Greece

E-mail address: `kouvid@uoc.gr`