Geometriae Dedicata **78:** 327–343, 1999.

On the Symmetric Product of a Curve with General Moduli

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(Received: 19 February 1999)

Abstract. We study the problem of describing the cone of the effective divisors in the second symmetric product of a curve with general moduli using a degeneration to a rational g-nodal curve.

Mathematics Subject Classifications (1991): 14H10, 14C20, 14J40.

Key words: curve with general moduli, effective divisor, symmetric product of curve.

0. Introduction

Let C be a smooth irreducible curve of genus $g \geqslant 1$. Its second symmetric product $C^{(2)}$ is a smooth variety and parametrizes unordered pairs of points of C. It is defined as the quotient of the ordinary product by the natural involution and the canonical map $\alpha: C \times C \to C^{(2)}$ is ramified along the diagonal. For $P, Q \in C$ we denote by $P + Q$ the corresponding point in $C^{(2)}$. On $C^{(2)}$, we can define 'natural' divisors in the following way. Fix a point P on C and define $X_P := \{P + Q,\ Q \in C\}$ the divisor associated to the point P. We write x for the class of X_P in the Neron–Severi group (which is independent from the choice of P). We also define $\Delta := \{2Q,\ Q \in C\}$ to be the diagonal divisor on $C^{(2)}$ and we write δ for its class. The diagonal divisor on $C \times C$ defines an invariant line bundle under the involution and so, it descends to a line bundle on $C^{(2)}$. Since the map α is ramified along the diagonal, the square of the latter bundle is isomorphic to Δ. We denote this bundle by $\Delta/2$.

For a curve C with general moduli, it is known that the Neron–Severi group of $C^{(2)}$ is generated over $\mathbb{Q}$ by the classes x and δ, see [4], Chapters 2 and 5, p. 282. Moreover, the class of a line bundle on $C^{(2)}$ is an integral combination of x and $\delta/2$. We write $\mathcal{L}_{n,\gamma}$ for a line bundle of class $(n + \gamma)x - \gamma(\delta/2)$. The number γ is called the valence of the line bundle (for its geometric meaning, see again [4]). Given $P \in C$, we denote by $\mathcal{L}_P$ the line bundle which corresponds to the divisor

* The second author would like to thank the University of Rome 2 for its hospitality.

X_P. Take a line bundle L on C and write it in the form $L = \mathcal{O}(\Sigma_i R_i - \Sigma_j Q_j)$, for some $R_i, Q_j \in C$. We then define on the symmetric product the line bundle $\mathcal{L}_L := \mathcal{O}(\Sigma_i X_{R_i} - \Sigma_j X_{Q_j})$. It turns out that every line bundle $\mathcal{L}_{n,\gamma}$ can be written in the form $\mathcal{L}_{n,\gamma} = \mathcal{L}_L - \gamma(\Delta/2)$, where L is a line bundle on C of degree $n + \gamma$, see [4].

The purpose of this work is to study the cone of the effective divisors on $C^{(2)}$ in the $x,\ \delta/2$-plane. We do this by using a degeneration of the symmetric product of the smooth curve of genus g to the symmetric product of a rational g-nodal curve given by Franchetta, see [3]. The problem then is reduced to a 'Nagata type' problem for plane curves of specific degree possessing singularities of specific order at a number of general points of the plane and satisfying some additional conditions. By using that degeneration we are able to reprove and improve some results from [6] and to reduce the general problem to the Nagata conjecture, see [7].

1. Franchetta's Degeneration

In this section we describe a degeneration of the symmetric product of a curve with general moduli given by Franchetta in [3]. We will do this in a way which is rather different from Franchetta's original one, and which seems to us more transparent, although our analysis leads exactly to the same results of Franchetta's. Suppose $\pi\colon X \to U$ is a flat family of curves over an open disk U with X smooth, $X_0 = \pi^{-1}(0)$ a rational g-nodal curve, and all other fibers of the map π smooth curves of genus g. We form the fibered symmetric product $p\colon Y \to U$ of the family, the fibers of which are the varieties parametrizing unordered pairs of points of the fibers of π. For $t \neq 0$, the fiber Y_t of p is the (smooth) symmetric product of the curve X_t, while the zero fiber Y_0 is singular. After taking the normalization $\nu\colon \tilde{X}_0 \to X_0$ of X_0, we have that the curve $\tilde{X}_0$ is a smooth rational curve and so, its symmetric product is isomorphic to $\mathbb{P}^2$. Under the isomorphism, the diagonal in the symmetric product of $\tilde{X}_0$ corresponds to a smooth conic Γ and the curves X_P, for $P \in \tilde{X}_0$, correspond to lines (which we denote by the same notation) tangent to Γ at the point $2P$. The symmetric product Y_0 of X_0 is then birationally equivalent to $\mathbb{P}^2$. If $P_1, \ldots, P_g$ are the nodes of X_0, then the birational map $\psi\colon Y_0 \cdots \to \mathbb{P}^2$ is defined outside of the union of the curves $X_{P_i},\ \ i = 1, \ldots, g$, and sends the point $P + Q$, where P, Q are smooth points of X_0, to the point in $\mathbb{P}^2$ corresponding to $P' + Q'$, where $P' := \nu^{-1}P$ and $Q' := \nu^{-1}Q$.

We describe in the following the local picture of the threefold Y and of its zero fiber Y_0 in a neighborhood of a point $P_i + Q \in X_{P_i}$, for $Q \in X_0$. We start with the case $Q = P_i$, $i = 1, \ldots, g$. In local coordinates, in a neighborhood of a node of the zero fiber, the family $\pi\colon X \to U$ is given by the equation $x_1x_2 = t$ in $\mathbb{C}^2 \times U$. We want to write the local equations of the threefold Y in a neighborhood of the point $2P_i$. For that, we first write the equations which define the variety $\mathrm{Sym}^2\mathbb{C}^2$. We start with the ordinary product $\mathbb{C}^2 \times \mathbb{C}^2$ which has coordinates

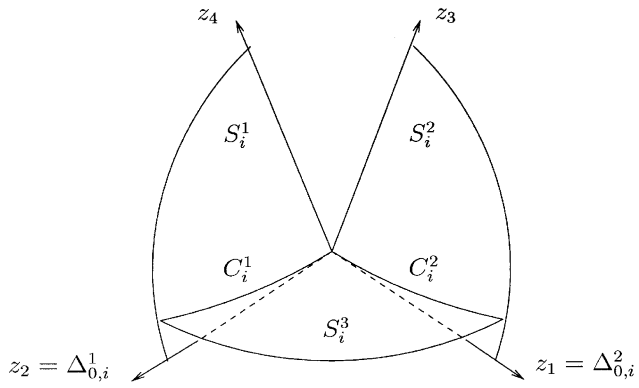

Figure 1. The local picture of Y at the origin.

$(x_1, x_2;\ y_1, y_2)$. Then $\mathrm{Sym}^2\mathbb{C}^2$ is given as the quotient of that by the equivalence relation $(x_1, x_2;\ y_1, y_2) \sim (y_1, y_2;\ x_1, x_2)$. After making a change of coordinates $\alpha\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ given by $s_i = (x_i + y_i)/2$ and $r_i = (x_i - y_i)/2$, we get in the new coordinate system $(s_1, s_2;\ r_1, r_2)$ that the above relation becomes $(s_1, s_2;\ r_1, r_2) \sim (s_1, s_2;\ -r_1, -r_2)$. Therefore $\mathrm{Sym}^2\mathbb{C}^2$ is isomorphic to the product $\mathbb{C}^2 \times (\mathbb{C}^2/\sim)$, where the equivalence relation $\sim$ on $\mathbb{C}^2$ is defined by $(r_1, r_2) \sim (-r_1, -r_2)$. The quotient $\mathbb{C}^2/\sim$ can be realized as a quadric cone in $\mathbb{C}^3$. Indeed, its coordinate ring is given by the degree 2 symmetric polynomials in r_1, r_2. These are generated by $z_3 = r_1^2, z_4 = r_2^2, z_5 = r_1 r_2$ with a single relation $z_5^2 = z_4 z_3$. In other words, if we let $\beta\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^3$ be the map given by $z_1 = s_1, z_2 = s_2, z_3 = r_1^2, z_4 = r_2^2, z_5 = r_1 r_2$, then the composition $\gamma := \beta\alpha$ factors through the $\mathrm{Sym}^2\mathbb{C}^2$ and embeds the latter in $\mathbb{C}^5$ as a singular hypersurface given by the product $\mathbb{C}^2 \times V$, where V is the cone in $\mathbb{C}^3$ with equation $z_5^2 = z_3 z_4$.

To determine now the equations for the symmetric product of the family $x_1 x_2 = t$, we take the fibered ordinary product of that family in $\mathbb{C}^2 \times \mathbb{C}^2 \times U$ given by $x_1 x_2 = t,\ y_1 y_2 = t$ and we find the image of that in $\mathbb{C}^5 \times U$ under the map $\gamma \times \mathrm{id}$. An easy calculation yields that the above image is defined by the system

$$z_5^2 - z_3 z_4 = 0, \qquad z_1 z_5 + z_2 z_3 = 0,$$
$$z_1 z_4 + z_2 z_5 = 0, \qquad z_1 z_2 + z_5 - t = 0.$$

One then verifies that these equations define a fibration of surfaces, over the disk, which we denote, again, by $p\colon Y \to U$, with the total space Y singular only at the origin. For $t \neq 0$, the fiber is a smooth surface, while for $t = 0$ the fiber is a union of three surfaces in $\mathbb{C}^5$ as pictured in Figure 1: the surface S_i^1 which is the $z_2 z_4$-plane (corresponding to the image of $x_1 = 0,\ y_1 = 0$, i.e., this is the symmetric product of the branch of X_0 of equation $x_1 = 0$), the surface S_i^2 which is the $z_1 z_3$-plane (corresponding to the image of $x_2 = 0,\ y_2 = 0$, i.e., this is the symmetric product of the branch of X_0 of equation $x_2 = 0$) and the surface S_i^3

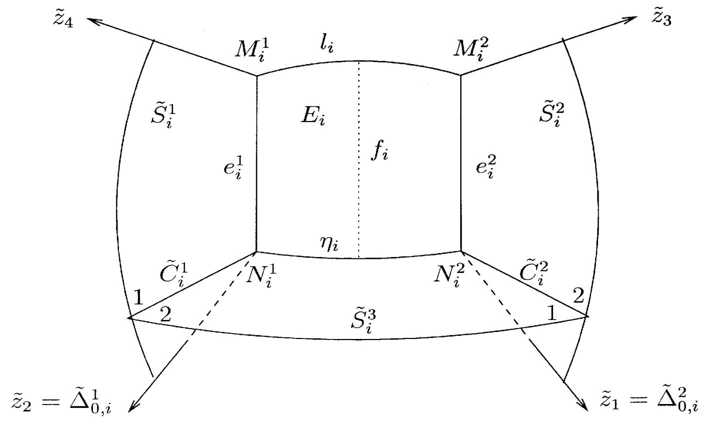

Figure 2. Local picture of the blowing up $\tilde{Y}$ of Y at the origin.

defined by the equations $z_3 = z_1^2$, $z_4 = z_2^2$, $z_5 = -z_1z_2$ (corresponding to the image of $x_1 = 0,\ \ y_2 = 0$ which is the same as that of $x_2 = 0,\ \ y_1 = 0$, i.e., this is the ordinary product of the branches of X_0 of equations $x_1 = 0$ and $x_2 = 0$). Note that S_i^1 and S_i^2 intersect only at the origin. The surface S_i^3 intersects S_i^1 along the curve $C_i^1\colon z_4 = z_2^2$ (in the z_2z_4-plane) and the surface S_i^2 along the curve $C_i^2\colon z_3 = z_1^2$ (in the z_1z_3-plane). The curves C_i^1, C_i^2 correspond to points in the symmetric product of X_0 of the form $P_i + Q$, $Q \in C$, i.e., they are pieces of the curve we have denoted by X_{P_i}. The diagonal $\mathcal{D}$ of the fibered symmetric product passes through the points $2P_i,\ \ i = 1, \ldots, g$, of the threefold Y. In a neighborhood of those points, $\mathcal{D}$ is defined by the equations $z_3 = z_4 = z_5 = 0,\ z_1z_2 = t$. The intersection Δ_0 of the diagonal with the zero fiber ($t = 0$) is therefore, in Figure 1, the union, $\Delta_{0,i}^1,\ \Delta_{0,i}^2$, of the z_2, z_1-axes respectively.

We will need to work with a fibration having a smooth total space. In order to do this, we blow up the origin in $\mathbb{C}^5 \times U$. The proper transform $\tilde{Y}$ of the threefold Y turns out to be smooth. The exceptional divisor is a $\mathbb{P}^5$ and its intersection E_i with $\tilde{Y}$ is given by the system

$$\lambda_5^2 - \lambda_3\lambda_4 = 0, \qquad \lambda_1\lambda_5 + \lambda_2\lambda_3 = 0,$$
$$\lambda_1\lambda_4 + \lambda_2\lambda_5 = 0, \qquad \lambda_5 - \lambda = 0,$$

where $\lambda_1, \ldots, \lambda_5, \lambda$ are the homogeneous coordinates for $\mathbb{P}^5$, see Figure 2. The above equations define an $\mathbb{F}_1$ surface. Its zero section l_i, having $l_i^2 = 1$, is given by the equations: $\lambda_1 = \lambda_2 = 0,\ \lambda_5^2 - \lambda_3\lambda_4 = 0,\ \lambda = \lambda_5$ and the infinity section η_i, having $\eta_i^2 = -1$, by the equations: $\lambda_3 = \lambda_4 = \lambda_5 = \lambda = 0$. We write e_i^1 and e_i^2 for the fibers of E_i passing through the points $M_i^1 = [000100]$ and $M_i^2 = [001000]$ of l_i. They are defined by the equations $\lambda_1 = \lambda_3 = \lambda_5 = \lambda = 0$ and $\lambda_2 = \lambda_4 = \lambda_5 = \lambda = 0$ respectively. The fibers e_i^1 and e_i^2 intersect the section at infinity η_i at the points $N_i^1 = [010000]$ and $N_i^2 = [100000]$ respectively.

The zero fiber $\tilde{Y}_0$ of the smooth threefold $\tilde{Y}$ is the union of the proper transforms $\tilde{S}_i^k$ of S_i^k, $k = 1, 2, 3$, and the exceptional divisor E_i. The components of $\tilde{Y}_0$ intersect as indicated in Figure 2. The surface $\tilde{S}_i^1$ (resp. $\tilde{S}_i^2$) intersects E_i along the fiber e_i^1 (resp. e_i^2). The surface $\tilde{S}_i^3$ intersects E_i along the infinity section η_i and the proper transform $\tilde{\mathcal{D}}$ of the diagonal does the same. The intersections of $\tilde{S}_i^3$, $\tilde{\mathcal{D}}$ and E_i along η_i are pairwise transverse. The meaning of the numbers 1, 2 in Figure 2 will be explained later.

To finish our local analysis at the points of the form $P_i + Q$, we describe (briefly) the local picture for the cases $Q \neq P_j$ and $Q = P_j$, for $j \neq i$. The first one is actually clear from the above analysis: such points correspond to points of C_i^1 or C_i^2 different from the origin; the threefold is smooth at those points but the central fiber Y_0 has a normal crossing double curve along the locus of those points. For the second, observe that a neighborhood of a point $P_i + P_j$, for $i \neq j$, in the symmetric product is isomorphic to a corresponding neighborhood at the point (P_i, P_j) in the ordinary product under the natural map. The local equations in the ordinary product are given in $\mathbb{C}^4 \times U$ by the system $x_1 x_2 = t,\ y_1 y_2 = t$. This variety has the origin as a singular point which we blow up. One can see that the proper transform $\tilde{Y}$ of Y intersects the exceptional divisor in a smooth quadric surface E_{ij} and the proper transform of the zero fiber Y_0 is a union of four surfaces which intersects E_{ij} along pairs of rulings a_{ij}^1, a_{ij}^2 and b_{ij}^1, b_{ij}^2 of each one of the two families of rulings of E_{ij}, see the first graph in Figure 4 below.

We are coming now to the global picture. The zero fiber Y_0 of the threefold $p\colon Y \to U$ is an irreducible surface birationally equivalent to $\mathbb{P}^2$ with singularities along the curves X_{P_i}, $i = 1, \dots, g$. After blowing up each one of the points $P_i + P_j$, $1 \leqslant i \leqslant j \leqslant g$, the zero fiber $\tilde{Y}_0$ consists of an irreducible surface $\tilde{S}$, which is the proper transform of Y_0, and a union of exceptional divisors E_i and E_{ij}, for $1 \leqslant i < j \leqslant g$, where E_i corresponds to the point $2P_i$ and E_{ij} to the point $P_i + P_j$, $i \neq j$. The surface $\tilde{S}$, intersects E_i in three exceptional curves η_i and e_i^1, e_i^2 and intersects E_{ij} in four exceptional lines a_{ij}^1, a_{ij}^2 and b_{ij}^1, b_{ij}^2. The surface $\tilde{S}$ is singular, with normal crossings along the curves $\tilde{X}_{P_i}$ corresponding to the proper transforms of the curves X_{P_i} in the zero fiber Y_0 of Y. The proper transform $\tilde{\mathcal{D}}$ of the diagonal, intersects $\tilde{Y}_0$ along the curve $\tilde{\Delta}_0$, which is the proper transform of the diagonal Δ_0 on Y_0, and along the exceptional curves η_i, $i = 1, \dots, g$.

To normalize the surface $\tilde{S}$, we unfold along the curves $\tilde{X}_{P_i}$. Let W be the resulting smooth surface and $\sigma\colon W \to \tilde{S}$ the corresponding map. Each curve $\tilde{X}_{P_i}$, has as preimage a pair of curves, which we denote by $\tilde{X}_{P_i^1}$ and $\tilde{X}_{P_i^2}$, $i = 1, \dots, g$. The birational map $\psi\colon Y_0 \cdots \to \mathbb{P}^2$, mentioned in the beginning of the section, lifts to a birational map $\phi\colon W \to \mathbb{P}^2$, which turns out to be regular. In the preimage of the singular locus of Y_0 in W, the map ϕ is described as follows. Let P_i^1, P_i^2 be the preimages of the singular points P_i, $i = 1, \dots, g$, under the normalization map $\nu\colon \tilde{X}_0 \to X_0$. The map ϕ contracts all the above exceptional curves as follows: the

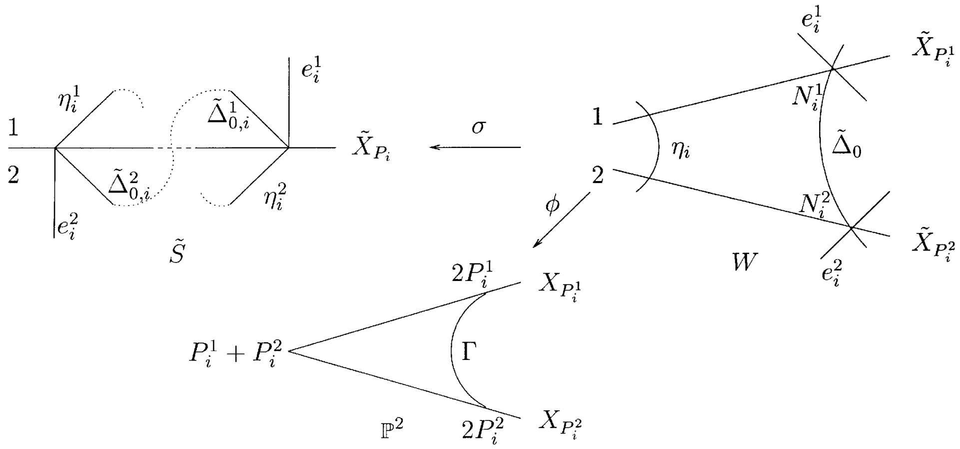

Figure 3. Normalization of $\tilde{S}$ at the point $2P_i$.

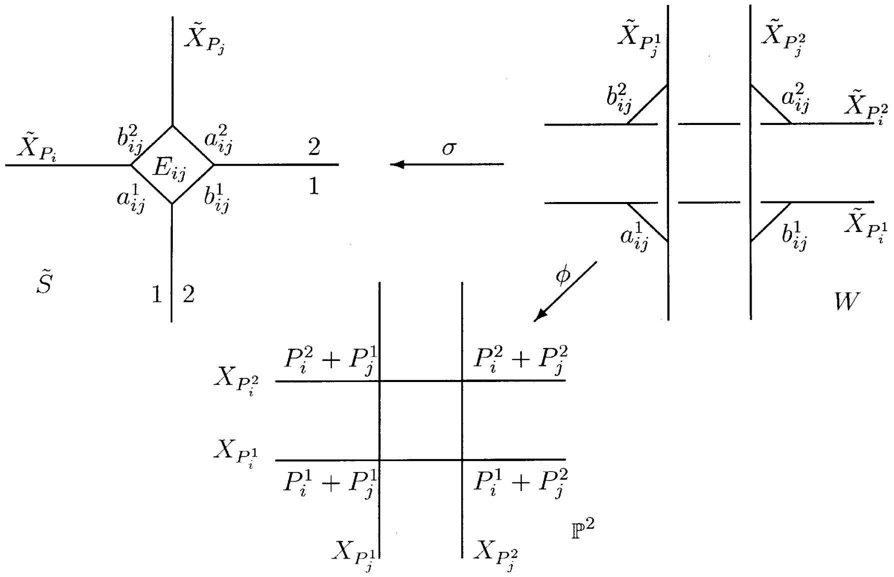

Figure 4. Normalization of $\tilde{S}$ at the point $P_i + P_j$, $i \neq j$.

curve η_i contracts to the point $P_i^1 + P_i^2$, the curves e_i^k to the point $2P_i^k$, $k = 1, 2$, and the curves $a_{ij}^1, a_{ij}^2, b_{ij}^1, b_{ij}^2$ contract to the points $P_i^1 + P_j^1$, $P_i^2 + P_j^2$, $P_i^1 + P_j^2$, $P_i^2 + P_j^1$ respectively. Also, the image of the curve $\tilde{X}_{P_i^1}$ (resp. $\tilde{X}_{P_j^2}$) is the line $X_{P_i^1}$ (resp. $X_{P_i^2}$) and the image of the curve $\tilde{\Delta}_0$ in $\tilde{Y}_0$ is the conic Γ.

The Figures 3 and 4 represent the normalization W of the surface $\tilde{S}$ and the map ϕ to the plane. Figure 3 corresponds to the singularity at the point $2P_i$. To explain that figure: the curve $\tilde{X}_{P_i}$ is a double curve in $\tilde{S}$ and along that curve the surface $\tilde{S}$ has two branches which intersect transversally. In Figure 2, the numbers '1, 2' refer to those two branches. The alternating appearance of those numbers is due to the fact that the symmetric product of a curve can be realized as the ordinary product 'folded' along the diagonal. The first graph in the Figure 3 corresponds to Figure 2, when one connects properly the two pieces $\tilde{C}_i^1$, $\tilde{C}_i^2$ to a single piece on the curve $\tilde{X}_{P_i}$. Figure 4 corresponds to a singularity at a point $P_i + P_j$, $i \neq j$. The situation in that figure is clear!

Before we continue, we describe one more geometric property of the above global picture. Since the curves $\tilde{X}_{P_i^1}$ and $\tilde{X}_{P_i^2}$ on W are produced from the unfolding of the surface $\tilde{S}$ along the curve X_{P_i}, there is a natural isomorphism between their points. There is then an induced isomorphism, which we denote by ω_i, between the points of the lines $X_{P_i^1}$ and $X_{P_i^2}$ of the plane. Geometrically, for each point $P \in \tilde{X}_0$, we take the tangent to the conic Γ at the point $2P$. This intersects the above pair of lines in a pair of points which correspond to each other under ω_i.

2. Limits of Line Bundles

In this section we continue our analysis of Franchetta's degeneration and we describe the limits of line bundles and sections, see Proposition 2.1 below. Suppose that we have a line bundle $\mathcal{L}$ defined on the total space Y outside of the zero fiber, i.e., a line bundle on $Y \backslash Y_0$. This induces a line bundle, which we denote again by $\mathcal{L}$, on $\tilde{Y} \backslash \tilde{Y}_0$. Since $\tilde{Y}$ is smooth, we can extend $\mathcal{L}$ to a line bundle $\bar{\mathcal{L}}$ on $\tilde{Y}$ and we want to describe its limit (i.e., its restriction) on the zero fiber $\tilde{Y}_0$. Note that we have many ways to extend $\mathcal{L}$ to $\bar{\mathcal{L}}$ on $\tilde{Y}$ (and so, $\mathcal{L}$ has many limits on $\tilde{Y}_0$), which differ by multiplication by a factor $\mathcal{O}(D)$, where D is a combination of the irreducible components of $\tilde{Y}_0$.

To specify a line bundle on $\tilde{Y}_0$ it suffices to determine its restriction on each one of the irreducible components of $\tilde{Y}_0$. On the surface $\tilde{S}$, the line bundles are described by their pull back to the smooth surface W. The Picard group of W is generated by the line bundle H, which is the pull back of the hyperplane line bundle of $\mathbb{P}^2$, and the line bundles η_i and e_i^1, e_i^2 defined by the exceptional lines (we use the same notation for the line bundles and the corresponding divisors). The Picard group of E_i is generated by the line bundle η_i corresponding to the section at infinity and the line bundle f_i corresponding to a fiber of the ruled surface, see again Figure 2. The Picard group of the surfaces E_{ij} is that of the surface $\mathbb{P}^1 \times \mathbb{P}^1$. Note then that on each component of $\tilde{Y}_0$, a line bundle is determined by its class in the Neron–Severi group. We are going to use the same notation for a line bundle or its class on those surfaces. To simplify things, we set $\eta = \Sigma_{i=1}^g \eta_i$, $f = \Sigma_{i=1}^g f_i$ and $e = \Sigma_{i=1}^g e_i^1 + e_j^2$ (equalities of divisors). We also write E for the (disjoint) union of the surfaces E_i, $i = 1, \dots, g$. Note that on E we have the equality $e = 2f$ in the Neron–Severi group. In what follows, we deal only with line bundles which are trivial on the surfaces E_{ij} and for which their restriction on W (i.e., their pull back from $\tilde{S}$) has class a which belongs to $\mathbb{Z}[H, \eta, e]$ and their restriction on E has class b which belongs to $\mathbb{Z}[\eta, f]$. We will say that such a line bundle is of type $[a, b]$.

To describe now the limits of $\mathcal{L}$ on the zero fiber $\tilde{Y}_0$, we assume that the class of the restriction of $\mathcal{L}$ to a fiber is given by $(n+\gamma)x-\gamma(\delta/2)$. Then $\mathcal{L}$ can be written in the form $\mathcal{L}_L-\gamma(\mathcal{D}/2)$, where L is a line bundle on $X\backslash X_0$ of degree $n+\gamma$ on the fibers of the map $\pi\colon X \to U$ and $\mathcal{D}/2$ is the line bundle on $\tilde{Y}\backslash\tilde{Y}_0$ which restricts to the line bundle $\Delta/2$ on the fibers. It suffices therefore to determine limits for the line bundles $\mathcal{L}_L$ and $\mathcal{D}/2$.

For the first. The line bundle L is extended in a unique way to a line bundle $\bar{L}$ on $X(X$ is smooth and X_0 is irreducible). We choose a meromorphic section s of $\bar{L}$ so that its support does not contain any of the singular points of the zero fiber X_0. Let $\Sigma_i R_i - \Sigma_j Q_j$ be the divisor of the restriction of s on X_0. We can take then as limit of $\mathcal{L}_L$ on $\tilde{Y}_0$ the line bundle which is $\mathcal{O}(\Sigma_i X_{R_i} - \Sigma_j X_{Q_j})$ on $\tilde{S}$ and which is trivial on the other components of $\tilde{Y}_0$. In terms of the normalization W of $\tilde{S}$, the line bundle is given by $(n+\gamma)H$ on W. Thus, we get as limit of $\mathcal{L}_L$ on $\tilde{Y}_0$ the one given by the data $[(n+\gamma)H,\ 0]$.

On the other hand, to determine a limit for $\mathcal{D}/2$, we first find a limit for $\mathcal{D}$ (the line bundle associated to the diagonal divisor $\mathcal{D}$). This is determined by the restriction on $\tilde{Y}_0$ of the proper transform $\tilde{\mathcal{D}}$ of the diagonal divisor on Y. We have seen that $\tilde{\mathcal{D}}$ cuts on $\tilde{S}$ the divisor $\tilde{\Delta}_0$ plus the divisor η and on E the divisor η. The pull back of the curve $\tilde{\Delta}_0$ on W is the proper transform of the conic Γ of the plane, see Figure 3. Its class is equal to $2H-e$. Therefore the resulting limit of $\mathcal{D}$ is given by the data $[2H-e+\eta,\ \eta]$. For the bundle $\mathcal{D}/2$, a limit is determined by the property that its square is equivalent (up to multiplication by components of the zero fiber) to $[2H-e+\eta,\ \eta]$. To find such a limit, we first replace the above limit of $\mathcal{D}$ by the new one which is obtained after taking the multiplication by the line bundle $\mathcal{O}(E)$. This is given by the data $[2H+2\eta,\ -2f]$. We then take as a limit for $\mathcal{D}/2$ the one given by $[H+\eta,\ -f]$.

The above choices of limits for $\mathcal{L}_L$ and $\mathcal{D}/2$ determine a unique way of extending $\mathcal{L} = \mathcal{L}_L - \gamma(\mathcal{D}/2)$ to a line bundle $\bar{\mathcal{L}}$ on $\tilde{Y}$ and the corresponding limit on the zero fiber is given by $[nH-\gamma\eta,\ \gamma f]$. From now on we fix those choices. In the following, we would like to describe limits of sections of $\mathcal{L}$. If $s_{n,\gamma}$ is a section of $\mathcal{L}$, then it can be extended, in a unique way, to a meromorphic section of the line bundle $\bar{\mathcal{L}}$. We denote by t the parameter on the disk U and we keep the same notation for the pull back on $\tilde{Y}$. We then multiply the above extended section by an appropriate (least) power of t in order to get a holomorphic section of the same bundle $\bar{\mathcal{L}}$, see [2] for a more detailed description of this procedure. We denote by $s^0_{n,\gamma}$ the restriction of that section on the zero fiber $\tilde{Y}_0$ and we write $D^0_{n,\gamma}$ for the corresponding divisor on $\tilde{Y}_0$. Note that $s^0_{n,\gamma}$ may vanish on some of the components of $\tilde{Y}_0$. To describe limits of sections, we have to separate cases:

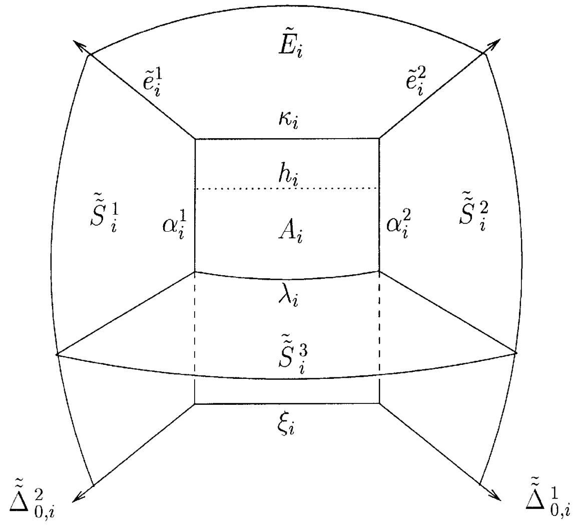

Figure 5. The blowing up $\tilde{\tilde{Y}}$ of $\tilde{Y}$ along the curve η_i.

$\gamma \geqslant 0$. We note that $D_{n,\gamma}^0$ cannot contain W. Otherwise, $D_{n,\gamma}^0$ would restrict on E to an effective divisor containing η. Since the class of this restriction on the other hand is γf, we see that $D_{n,\gamma}^0$ also would contain E and therefore the whole fiber; a contradiction. Therefore, $D_{n,\gamma}^0$ defines on W a curve C_W, with class $nH - \gamma\eta$. Its image, $C_{\mathbb{P}^2}$, to the plane is going therefore to be a curve of degree n passing with multiplicity γ through the g points $P_i^1 + P_i^2$, $i = 1, \ldots, g$. There is an additional property that $C_{\mathbb{P}^2}$ satisfies. The curve C_W is the preimage of a curve on the surface $\tilde{S}$. Therefore the curve $C_{\mathbb{P}^2}$ must restrict on the curves $X_{P_i^1}$ and $X_{P_i^2}$ at points which correspond to each other under ω_i. Note that this condition implies that if the curve passes through one point of the quadruple $P_i^1 + P_j^1$, $P_i^2 + P_j^2$, $P_i^1 + P_j^2$, $P_i^2 + P_j^1$ then passes also through the remaining three points. through those γ points.

$\gamma < 0$. We write $\gamma = -2\gamma'$, $\gamma' > 0$ (resp. $\gamma = -(2\gamma' + 1)$). The class of the line bundle is given by $[nH + 2\gamma'\eta, -2\gamma' f]$ (resp. $[nH + (2\gamma' + 1)\eta, -(2\gamma' + 1)f]$). This implies that $s_{n,\gamma}$ must vanish on the divisor E with multiplicity γ' (resp. $\gamma' + 1$). But then the restriction of $D_{n,\gamma}^0$ on $\tilde{S}$ must contain the divisor $\gamma'\eta + \gamma' e$ (resp. $(\gamma' + 1)\eta + (\gamma' + 1)e$). In other words, $D_{n,\gamma}^0$ does not vanish on W and induces on it a curve C_W with the property $C_W = C_W' + \gamma'\eta + \gamma' e$ (resp. $C_W = C_W' + (\gamma' + 1)\eta + (\gamma' + 1)e$), for some curve C_W' of class $nH - \gamma' e + \gamma'\eta$ (resp. $nH - (\gamma' + 1)e + \gamma'\eta$). The image on the plane of the curve C_W, which is the same as that of C_W', is going then to be a curve $C_{\mathbb{P}^2}$ of degree n passing with multiplicity γ' (resp. $\gamma' + 1$) through each one of the $2g$ points $2P_i^1$, $2P_i^2$, $i = 1, \ldots, g$. It satisfies also a similar property as in the positive valence case with respect to ω_i.

Moreover, in the case of negative valence there are additional properties satisfied by the 'limiting' curves $C_{\mathbb{P}^2}$. This is, basically, due to the fact that the exceptional divisors which are involved in this case are the e_i^1's and e_i^2's and these belong to the same branches of $\tilde{S}$, meeting along the curves $\tilde{X}_{P_i}$, together with the diagonal divisor $\tilde{\Delta}_0$. On the other hand, in the positive valence case, the exceptional divisors which are involved are the η_i's and these do not belong to the same branches of $\tilde{S}$ together with the diagonal, see the first graph in Figure 3. To understand those properties, we need to make a more detailed analysis of our blowing up picture.

In the smooth threefold $\tilde{Y}$, we blow up the curves η_i. In the proper transform $\tilde{\tilde{Y}}$ of $\tilde{Y}$, the exceptional divisors A_i are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Note that the zero fiber of the map $\tilde{\tilde{Y}} \to U$ contains the divisors A_i with multiplicity two. The proper transforms $\tilde{E}_i$ of E_i, $\tilde{\tilde{S}}_i^3$ of $\tilde{S}_i^3$ and $\tilde{\tilde{\mathcal{D}}}$ of $\tilde{\mathcal{D}}$ intersect the exceptional divisor A_i along the (disjoint) curves κ_i, λ_i and ξ_i respectively. The proper transforms $\tilde{\tilde{S}}{}_i^1$ of $\tilde{S}_i^1$ and $\tilde{\tilde{S}}_i^2$ of $\tilde{S}_i^2$ intersect the exceptional divisor A_i along the curves α_i^1 and α_i^2 respectively. We put, in general, (extra) tildes to denote proper transforms (of proper transforms!). Figure 5, hopefully, explains the situation better.

Note that there is a natural isomorphism, $\omega_i^{'}$, between α_i^1 and α_i^2 which is defined by the natural identification of the fibers of the $\mathbb{P}^1 \times \mathbb{P}^1$ surface A_i. The geometric meaning of that isomorphism is the following. The points of the curve α_i^1 (resp. α_i^2) correspond to the tangent directions at the point N_i^1 (resp. N_i^2) of the surface W, see Figure 3. These in turn, correspond to the second order jets tangent to the line $X_{P_i^1}$ (resp. $X_{P_i^2}$) at the point $2P_i^1$ (resp. $2P_i^2$) of the plane. Therefore, the isomorphism $\omega_i^{'}$ induces an isomorphism (we write again $\omega_i^{'}$) between the spaces of second second order jets at the points $2P_i^1$, $2P_i^2$ tangent to the lines $X_{P_i^1}$, $X_{P_i^2}$ respectively.

We define $\alpha := \Sigma_{i=1}^g \alpha_i^1 + \alpha_i^2$ and we make a similar definition for the κ, λ, ξ. We also define A to be the disjoint union of all the A_i's and we denote by h_i the class of the horizontal fibers of the surface A_i. We define $h_A := \Sigma_{i=1}^g h_i$. We use a triple to indicate the restriction of a line bundle on the surfaces $\tilde{\tilde{S}}$, $\tilde{E}$, A.

We start now with the line bundle $\mathcal{L}$ on $Y \backslash Y_0$. This induces a line bundle (we write again) $\mathcal{L}$ on $\tilde{\tilde{Y}} \backslash \tilde{\tilde{Y}}_0$ and we want to find its limits on $\tilde{\tilde{Y}}_0$. As before, it suffices to determine limits for the line bundles of the form $\mathcal{L}_L$ and $\mathcal{D}/2$. We omit most of the details (similar as above) and we come to the result. The bundle $\mathcal{L}_L$ has a limit of type $[(n + \gamma)H, 0, 0]$. On the other hand the limit of the bundle $\mathcal{D}$ is given by $[2H - 2\alpha - \tilde{e}, 0, \xi]$. We multiply by $\mathcal{O}(E)$ and we get the new limit $[2H - 2\alpha, -\tilde{e} - 2\kappa, \xi + \kappa]$. The appearance of the factor 2 in κ is due to the fact that A is counted with multiplicity two in the zero fiber of the map $\tilde{\tilde{Y}} \to U$. We can take therefore as limit of the bundle $\mathcal{D}/2$ the one given by the data $[H - \alpha, -\tilde{f} - \kappa, h_A]$. We fix those choices of the limits.

Before we continue our analysis we make a definition. We say that a plane curve has at the point $2P_i^k$, $k = 1, 2,\ i = 1, \dots, g$, singularity of type (a, b), $a \geqslant b$, if it has multiplicity a at the point $2P_i^k$ on the plane and, also, its proper transform has multiplicity b at the point N_i^k of W, see Figure 3. The point N_i^k of W is the point of intersection of the exceptional line e_i^k with the curve $\tilde{X}_{P_i^k}$, i.e., it is the point at infinity at $2P_i^k$ corresponding to the direction of the line $X_{P_i^k}$ in the plane. Note that such a curve has a tangent lines at the point $2P_i^k$, b of which coincide with the line $X_{P_i^k}$. Therefore, such a curve determines also b second order jets tangent to the line $X_{P_i^k}$ at the point $2P_i^k$.

Say first that $\gamma = -2\gamma', \gamma' > 0$. Then the limit of $\mathcal{L}$ on $\tilde{\tilde{Y}}_0$ is given by $[nH - 2\gamma'\alpha,\ -2\gamma'(\tilde{f} + \kappa),\ 2\gamma' h_A]$. Keeping the same notation as above, the section $s_{n,\gamma}$ must vanish on the divisor $\tilde{E}$ with multiplicity γ'. But then the restriction of $D^0_{n,\gamma}$ on $\tilde{\tilde{S}}$ must contain the divisor $\gamma'\tilde{e}$. In other words, $D^0_{n,\gamma}$ induces on $\tilde{W}$ a curve $C_{\tilde{W}}$ with the property $C_{\tilde{W}} = C'_{\tilde{W}} + \gamma'\tilde{e}$ for some curve $C'_{\tilde{W}}$ of class equal to $nH - 2\gamma'\alpha - \gamma'\tilde{e}$. Observe, also, that the curve $C_{\tilde{W}}$ intersects the curves $\alpha_i^1,\ \alpha_i^2$ in points which correspond to each other under ω_i'. This is due to the fact the restriction of the limit line bundle on the surface A_i has as a fixed part the divisor $\gamma'\kappa$ and a moving part of class $\gamma' h_A$. Therefore the image, $C_{\mathbb{P}^2}$, of the curve $C_{\tilde{W}}$ to the plane is going to be a curve of degree n having at the $2g$ points $2P_i^1,\ 2P_i^2$, $i = 1, \dots, g$, singularity of type $(\gamma',\ \gamma')$. Moreover, the γ' second order jets tangent to the line $X_{P_i^1}$ at the point $2P_i^1$, which are determined by that curve, correspond to those at the point $2P_i^2$ tangent to the line $X_{P_i^2}$ under the isomorphism ω_i'.

The corresponding result for the case $\gamma = -(2\gamma' + 1)$ is the following. In this case the curve $C'_{\tilde{W}}$ has class equal to $nH - (2\gamma' + 1)\alpha - (\gamma' + 1)\tilde{e}$. Note that the intersection of C'_W with each exceptional line α_i^k, $k = 1, 2$, is equal to γ'. Therefore the image, $C_{\mathbb{P}^2}$, of the curve $C_{\tilde{W}}$ on the plane is going to be a curve of degree n having at the $2g$ points $2P_i^1,\ 2P_i^2$, $i = 1, \dots, g$, singularity of type $(\gamma' + 1,\ \gamma')$. The curve $C_{\mathbb{P}^2}$ satisfies also a similar property with respect to the isomorphism ω_i' as above.

The following proposition summarizes the above results.

PROPOSITION 2.1. *The above described limit of a section of a line bundle $\mathcal{L}_{n,\gamma}$ on $Y \backslash Y_0$, induces on the plane $\mathbb{P}^2$ a curve $C_{\mathbb{P}^2}$ of degree n, which satisfies the following properties.*

(1) *If $\gamma \geqslant 0$, then the curve $C_{\mathbb{P}^2}$ passes through the g points $P_i^1 + P_i^2$, $i = 1, \dots, g$, with multiplicity γ and intersects the lines $X_{P_i^1}$ and $X_{P_i^2}$ in points which correspond to each other under ω_i.*

(2) *If $\gamma < 0$, then in the case $\gamma = -2\gamma', \gamma' > 0$ (resp. $\gamma = -(2\gamma' + 1)$) the curve $C_{\mathbb{P}^2}$ has at the $2g$ points $2P_i^1,\ 2P_i^2$, $i = 1, \dots, g$, singularity of type*

(γ', γ') (resp. $(\gamma'+1, \gamma')$). Moreover, the γ' second order jets tangent to the line $X_{P_i^1}$ at the point $2P_i^1$, which are determined by that curve, correspond to those at the point $2P_i^2$ tangent to the line $X_{P_i^2}$ under the isomorphism ω_i'. Also, the curve $C_{\mathbb{P}^2}$ intersects the lines $X_{P_i^1}$, $X_{P_i^2}$ in points which correspond to each other under ω_i.

3. Effective Divisors on the Symmetric Product

In this section we return to the problem stated in the Introduction. Let C be a curve of genus $g \geqslant 1$, with general moduli and $C^{(2)}$ its second symmetric product. We would like to describe the (rational) cone of the effective divisors on the smooth surface $C^{(2)}$ in the x, $\delta/2$-plane. We write $c_{n,\gamma}$ for the class of the line bundle $\mathcal{L}_{n,\gamma}$. In other words, the x-coordinate of the point $c_{n,\gamma}$ is $n+\gamma$ and the $\delta/2$-coordinate is $-\gamma$. The following basic intersection rules take place in the symmetric product: $x^2 = 1$, $x \cdot \delta = 2$, $\delta^2 = 4-4g$. We then have $c_{n,\gamma} \cdot c_{n',\gamma'} = nn' - g\gamma\gamma'$ and so, $c_{n,\gamma}^2 = n^2 - g\gamma^2$. The sign of the self intersection remains fixed for classes supported on the same ray. We can then talk for rays of positive, negative or zero self intersection.

If $c_{n,\gamma}$ is an effective class then $n > 0$. This is based on the fact that the divisor class x is an ample class, see ([1], Chapter VII, Proposition 2.2, p. 310), and so, $c_{n,\gamma} \cdot x = n > 0$. Also, one can easily see that for such a class we have $n+\gamma \geqslant 0$ (take the intersection with Δ). Therefore the effective cone is located on the right side of the $\delta/2$-axis.

A ray in the right half plane is determined by its slope, i.e., by the ratio of the $\delta/2$-coordinate by the x-coordinate of a point on the ray. If a class $c_{n,\gamma}$ is supported on a ray then the slope of that ray is $-\gamma/(n+\gamma)$. Given a ray R, of slope $-\gamma/(n+\gamma)$, supporting a class $c_{n,\gamma}$, there is an 'orthogonal' ray $R^\perp$, of slope $-n/(g\gamma+n)$, supporting the classes c_{n_0,γ_0} with $c_{n,\gamma} \cdot c_{n_0,\gamma_0} = 0$. The ray $R^\perp$ separates the right half plane in two parts. The sign of the intersection of the classes belonging in the one of those parts with $c_{n,\gamma}$ remains fixed.

We determine now the 'positive' cone, i.e., the cone of rays of positive self intersection. The boundary of that cone is given by the two rays determined by the equation $n = \pm\sqrt{g}\,\gamma$. We denote by R_1 and R_2 those rays (R_1 corresponds to the positive sign). Ray R_1 has slope $-1/(\sqrt{g}+1)$ and lies on the fourth quarter while ray R_2 has slope $1/(\sqrt{g}-1)$ and lies on the first quarter, see Figure 6. The claim is that each ray, with rational slope, in the interior of the positive cone supports an effective divisor class. Indeed, for such a class $c_{n,\gamma}$ we have $c_{n,\gamma}^2 > 0$ and $c_{n,\gamma} \cdot x > 0$. This implies that a multiple of that class is effective, see ([5], Chapter V, Corollary 1.8, p. 363). Therefore, the (rational) open positive cone is contained in the effective cone.

To continue our analysis, we have to determine the two boundary rays of the effective cone and then to examine if those rays belong or not to the cone, i.e., if

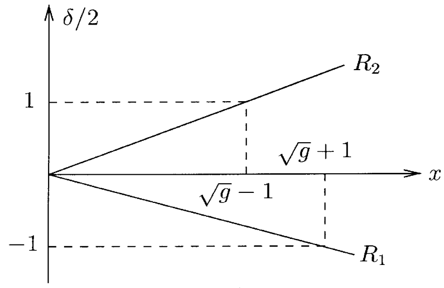

Figure 6. The x, $\delta/2$-plane.

those rays determine an open or a closed boundary. From the above discussion, a boundary ray will be a ray of self intersection $\leqslant 0$. Moreover, if we are able to find an *irreducible* divisor of self intersection $\leqslant 0$, this will determine a closed boundary for the cone. Indeed, let $c_{n,\gamma}$ be the class of such a divisor, say in the fourth quarter (resp. first quarter). If R is the ray supporting $c_{n,\gamma}$ then the orthogonal ray $R^{\perp}$ lies in the closure of the positive cone in the fourth quarter (resp. first quarter). Since for any irreducible divisor, different from the above, of class $c_{n',\gamma'}$ we have that $c_{n,\gamma} \cdot c_{n',\gamma'} = nn' - g\gamma\gamma' \geqslant 0$, this yields that the class of D lies on a ray which is located 'above' (resp. 'below') the ray $R^{\perp}$ and this implies the claim. On the other hand, the only case where the cone can possibly have an open boundary is when this boundary is one of the two rays R_1, R_2. Note, also, that by the above remarks we have the following: if $c_{n,\gamma}$, $c_{n',\gamma'}$ are classes of negative self intersection which both lie in the fourth quarter (or in the first quarter) then $c_{n,\gamma} \cdot c_{n',\gamma'} < 0$.

By the above observation, we can easily see that one side of the cone is closed. The (irreducible) diagonal divisor Δ has class $c_{2,-2}$ and so, its self intersection is given by $c_{2,-2}^2 = 4 - 4g \leqslant 0$. Therefore, the $\delta/2$-axis is the boundary of the cone on one side. To determine the boundary on the other side, which lies in the fourth quarter, we notice that the above discussion tells us this: either there exists an irreducible divisor D of class $c_{n,\gamma}$, $\gamma \geqslant 0$, and $c_{n,\gamma}^2 \leqslant 0$, in which case the corresponding ray is the other closed boundary of the cone, or the cone is open and the open boundary is the ray R_1. For low genera, one can construct such divisors D with $D^2 \leqslant 0$, see [6]. The construction is based on the following: given $C \to \mathbb{P}^1$ a $d:1$ map, then we can construct a divisor in the symmetric product $C^{(2)}$ as follows: to each point $P \in \mathbb{P}^1$, we associate in $C^{(2)}$ the $\binom{d}{2}$ points which correspond to the unordered couples of the d points of the fiber over p. By moving P in $\mathbb{P}^1$ we form a curve, Γ_d, in $C^{(2)}$. It turns out that for a 'general' such map (more specifically: a map with simple ramification) the above curve is smooth, irreducible and its class is given by $c_{d-1,1}$. We state now the results for low genera, see [6]:

(1) For $g = 1$. The elliptic curve C is hyperelliptic. The boundary of the cone in the fourth quarter is then determined by the ray supporting a Γ_2. This has class $c_{1,1}$ and $c_{1,1}^2 = 0$.
(2) For $g = 2$. The curve C is, again, hyperelliptic. In this case the Γ_2 has, also, class $c_{1,1}$ and self intersection $c_{1,1}^2 = -1$.
(3) For $g = 3$. Things are more complicated here, but again there is an irreducible divisor of class $c_{10,6}$ and of negative self intersection $c_{10,6}^2 = -8$. We will return to this later.
(4) For $g = 4$. The curve C possesses two g_3^1's. The corresponding Γ_3 determines the boundary since the self intersection is $c_{2,1}^2 = 0$.

To continue our study for higher genera, we want to use Franchetta's degeneration. To be able to do that, we make the following observation: if D is an irreducible effective divisor of class $c_{n,\gamma}$ $(\gamma \geqslant 0)$ and self intersection < 0, then it is unique with that property. Indeed, if we assume that D' is a different such irreducible divisor, then $D \cdot D' < 0$; a contradiction. Suppose now that a curve with general moduli possesses such a divisor. We can then take a family of curves $X \to U$, as in Section 1, with the smooth fibers to be curves of genus g with general moduli and the zero fiber a rational g-nodal curve. On each such fiber we can find a divisor as above, and by the uniqueness, those divisors fit together and form a divisor $\mathcal{N}$ in $X \backslash X_0$. We denote the associated line bundle by $\mathcal{L}_{n,\gamma}$. On the other hand, if we suppose that there exist on the symmetric product of a curve with general moduli an irreducible divisor D of class $c_{n,\gamma}$ $(\gamma \geqslant 0)$ and self intersection $= 0$, then there are finitely many such divisors. We take the sum of those, which defines a uniquely determined divisor and we continue as before.

We take now the limit of $\mathcal{L}_{n,\gamma}$ on the zero fiber in the way which was described in Section 2. The limit on the zero fiber of the section corresponding to the divisor $\mathcal{N}$, induces on the plane $\mathbb{P}^2$ a curve, $C_{\mathbb{P}^2}$, of degree n passing through the g points $P_i^1 + P_i^2$ with multiplicity γ (notation as in Section 2). And now, comes into the picture the Nagata conjecture (for equal multiplicities), which we state:

THE NAGATA CONJECTURE. *Let $Q_1, \ldots, Q_g$ be g points in general position on the plane $\mathbb{P}^2$, with $g \geqslant 10$. If there exist a curve C of degree n which passes through each one of the above points with multiplicity γ, then $n > \sqrt{g}\,\gamma$.*

The above conjecture has been proven, by Nagata, for the case where g is a perfect square ($g = m^2$, $m \in \mathbb{Z}$) but remains, still, open for the remaining cases, see [7].

To return now to our problem, we state the following proposition

PROPOSITION 3.1. *If g is a perfect square, with $g \geqslant 9$, then the boundary of the effective cone in the fourth quarter is given by the R_1 line, which is an open boundary line.*

Proof. We keep the above notation. We first show that we can choose the degeneration so that the points $P_i^1 + P_i^2$, $i = 1, \ldots, g$, are general points of the plane. This can be done easily: we start with g general points $Q_1, \ldots, Q_g$ in the plane and we choose a conic Γ containing none of the above points. Take the tangents $X_i^1,\ X_i^2$ from Q_i to Γ and let $A_i^1,\ A_i^2$ be the intersections with Γ. We choose as the rational g-nodal curve the one which is constructed from Γ by gluing the g pairs of points $A_i^1,\ A_i^2$. After degenerating to that curve, we have that $P_i^1 + P_i^2 = Q_i$ and $X_{P_i^k} = X_i^k$, $k = 1, 2$, $i = 1, \dots, g$. Note also, that we have the freedom in the choice of Γ. Observe now that the class $c_{n,\gamma}$ $(\gamma \geqslant 0)$ has self intersection $\leqslant 0$ if and only if $n \leqslant \sqrt{g}\,\gamma$. For $g \geqslant 10$ and g a perfect square, the Nagata result [7] which we mentioned above, tells us that, if $n \leqslant \sqrt{g}\gamma$ then there is no curve of degree n passing through the g points $P_i^1 + P_i^2$ with multiplicity γ and so, there is no possible limit curve corresponding to a divisor of class $c_{n,\gamma},\ \gamma \geqslant 0$ and $c_{n,\gamma}^2 \leqslant 0$. Therefore the (open) boundary of the cone is given by the R_1 line and this proves the proposition for $g \geqslant 10$.

On the other hand, one can give, in our case, a direct proof of the above proposition which covers also the case $g = 9$. For that, we use the following lemma

LEMMA 3.1. *Let $g = m^2$, $m \geqslant 3$ an integer. Let M be a smooth irreducible curve of degree m and let $Q_1, \dots, Q_g$ be $g = m^2$ points on M in general position. If a curve N, of degree $n = m\gamma$, passes through the points $Q_1, \dots, Q_g$ with multiplicity γ then N must be equal to γM.*

Proof. We write $N = aM + C$, where $a \leqslant \gamma$ and C is a curve of degree $k = m(\gamma - a)$, which does not contains M. The curve C intersects M at the points Q_i, $i = 1, \dots, g$, and passes through those points with multiplicity $\gamma - a$. Suppose now that $a < \gamma$; we are going to reach a contradiction. Let L_0 be the line bundle of degree $(\gamma - a)g = (\gamma - a)m^2$ on M, which is defined by the $(\gamma - a)m$ multiple of the hyperplane section H on M. Let $p = [(m-1)(m-2)]/2$ be the genus of M; note that $p \geqslant 1$ since $m \geqslant 3$. We denote by L_i, $i = 1, \dots, (\gamma - a)^{2p}$, the line bundles of degree m^2 on M, satisfying the property $L_i^{\otimes(\gamma - a)} = L_0$. We consider the Abel–Jacobi map $u\colon M^{(g)} \to J^g(M)$ from the symmetric product of M of degree $g = m^2$ to the Jacobian of the same degree. Then the existence of the above curve C, yields that the divisor $D := Q_1 + \cdots + Q_g$ lies on a fiber of the map u over one of the points L_i. This contradicts the genericity of the points Q_i. $\square$

We return now to the proof of our proposition: We make a degeneration to a rational g nodal curve, so that the $g = m^2$ points $P_i^1 + P_i^2$ to be the points Q_i, $i = 1, \dots, g$, of the curve M as in the lemma. This is possible by the above remarks. It is enough to show then that there is no limit curve $C_{\mathbb{P}^2}$ for the above configuration of points since this will imply the same for the generic configuration (by the upper semicontinuity of the dimension of linear systems). But if such a curve exists, then by the above lemma, it must be equal to γM. On the other hand, since we have the freedom in the choice of the conic Γ, the (fixed) curve γM does not satisfy the additional properties with respect to the isomorphisms ω_i and so, it can not be a limit curve. $\square$

As an outcome of the above discussion, we can also state the following more general (!) Proposition

PROPOSITION 3.2. *If the Nagata conjecture is true then, for $g \geqslant 9$, the boundary of the effective cone in the fourth quarter is given by the R_1 line, which is an open boundary line*.

For the remaining cases of low genera, not covered by the Nagata conjecture and by the above stated results of [6] i.e., for $5 \leqslant g \leqslant 8$, we were not able to find the boundary of the cone, although we expect to be the (open) R_1 line. In this range of genera, there is always a plane curve which violates the Nagata conjecture. For example, for $g = 5$ take the conic through five general points. This yields the existence of linear systems in the plane, with $n < \sqrt{g}\,\gamma$, and of big dimension (quadratic on γ). On the other hand the extra properties satisfied by the limit curves impose conditions which are linear on γ. Therefore, we have not been able to conclude anything from Franchetta's degeneration.

We finish with the picture for the case $g = 3$ and the corresponding degeneration: As we stated before, there is a divisor N of class $c_{10,6}$ with self intersection -8. This can be constructed, in at least, two ways: for the first, see [6]. For the second, we observe that the symmetric product $C^{(2)}$ of a generic curve of genus 3 has a natural involution. Indeed, if $D \in C^{(2)}$, then $h^0(D) = 1$ (by the genericity of the curve) and so, $h^0(K - D) = 1$, where K is the canonical bundle on C. We then define the involution by sending the point D to the point $K - D$ on $C^{(2)}$. We define now the divisor N on $C^{(2)}$ to be the image of the diagonal divisor Δ under the involution. Therefore N has the same self intersection as the diagonal, i.e., $N^2 = -8$. We degenerate to a rational three nodal curve. The associated plane curve $C_{\mathbb{P}^2}$ is the following. Let L_{ij} be the line joining the point $P_i^1 + P_i^2$ with $P_j^1 + P_j^2$, $1 \leqslant i < j \leqslant 3$. Then $C_{\mathbb{P}^2}$ is the union of twice those three lines plus a quartic curve Q which passes through the above points with multiplicity two. More specifically, take the blowing up plane $\tilde{\mathbb{P}}^2$ at the three points $P_i^1 + P_i^2$, $i = 1, 2, 3$. Let η_i, $i = 1, 2, 3$, be the exceptional divisors of the blow up. The plane $\tilde{\mathbb{P}}^2$ can be viewed as the blowing down of the surface W of Section 1, at the lines e_i^1 and e_i^2, $i = 1, 2, 3$, see Figure 3. We have seen that the limit of the diagonal line bundle, has on W class $2H + 2\eta$. The section in that bundle, corresponding to the diagonal divisor induces then on C_W, and consequently on $\tilde{\mathbb{P}}^2$, a curve of class $2H + 2\eta$. This is the union of the pull back of the conic Γ plus twice the exceptional divisors η_i, $i = 1, 2, 3$. The image of that union of curves under the 'Cremona transform', centered at the three points $P_i^1 + P_i^2$, $i = 1, 2, 3$, is the curve $C_{\mathbb{P}^2}$. It is a nice coincidence that the extra conditions with respect to the isomorphisms ω_i are automatically satisfied. In other words, the involution in $C^{(2)}$ corresponds, under the Franchetta degeneration, to the Cremona transform in the plane!

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