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On some results of Morita and their application to questions of ampleness

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Abstract. We give an algebro-geometric proof of some results of Morita concerning the pull back of a cohomology class in the Jacobian fibration of a family of curves to the family itself. We apply those to the question of ampleness of line bundles on the family of curves and its symmetric products.

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1 Introduction

Let $\pi : \mathcal{C} \longrightarrow B$ be a flat family of smooth curves over a smooth base B and let $\psi : \mathcal{J}^0 \longrightarrow B$ be the corresponding fibration of Jacobians. On $\mathcal{J}^0$ there is a closed 2-form η_0 whose restriction to each fiber is a representative of the class of the principal polarization and whose restriction to the zero section is trivial. The cohomology class of this form takes values in the rationals but, as it turns out, the class of the form $\eta = 2\eta_0$ takes values in the integers. In his papers [8], [9], Morita gives an expression of the pull back, via some canonical maps, of the cohomology class $[\eta]$ to the family of curves. In this note we give an algebro-geometric proof of Morita's results, see also [3] for another algebro-geometric approach. We do this by stating the above results on the level of line bundles, i.e., we define a line bundle whose chern class is given by $[\eta]$. We then give an application to the question of ampleness of line bundles on the above family of curves and its symmetric products.

2 Some auxiliary results

2.1 Line bundles on the symmetric product of a curve

Let C be a smooth irreducible curve of genus g. Its dth symmetric product $C_{(d)}$ is a smooth variety and parametrizes unordered d-tuples of points of C. It is defined as the quotient of its ordinary dth product $C^{\times d}$ by the natural action of the symmetric group of d elements. Let $q : C^{\times d} \longrightarrow C_{(d)}$ denote the canonical map, which is ramified along the diagonals. For $P_i \in C,\ i = 1, \dots, d$, we denote by $P_1 + \cdots + P_d$ the corresponding point in $C_{(d)}$. On $C_{(d)}$, we define natural divisors and line bundles in the following way. We define $\varDelta := \{D + 2Q,\ D \in C_{d-2},\ Q \in C\}$ to be the diagonal divisor on $C_{(d)}$. We keep the same notation for the corresponding line bundle. The sum of the diagonal divisors $\varDelta_{ij}$ on $C^{\times d}$ defines an invariant line bundle under the action of the symmetric group and so, it descends to a line bundle on $C_{(d)}$. Since the map q is ramified along the diagonals, the square of the latter bundle is isomorphic to $\varDelta$. We denote this bundle by $\varDelta/2$. Also, by fixing a point P on C, we define $X_P := \{D + P,\ D \in C_{(d-1)}\}$ the divisor associated to the point P. We denote by $\mathcal{L}_P$ the line bundle associated to the divisor X_P. We denote by x the class of $\mathcal{L}_P$ in the Neron-Severi group, which turns out to be independent from the choice of P. More generally, by taking an effective divisor $D = \sum_i P_i$ on C, we define on $C_{(d)}$ the divisor $X_D = \sum_i X_{P_i}$. Moreover, let M be a line bundle on C. We write $M = \mathcal{O}(D_1 - D_2)$, for some effective divisors D_1, D_2 on C and we then define $\mathcal{L}_M$ to be the line bundle $\mathcal{L}_M := \mathcal{O}(X_{D_1} - X_{D_2})$ on $C_{(d)}$. The proof of the following proposition is left to the reader.

Proposition 2.1. *If A, B are line bundles on C and $n, m \in \mathbb{Z}$, then we have that $n\mathcal{L}_A = m\mathcal{L}_B$ on $C_{(d)}$, if and only if, $nA = mB$.*

Proposition 2.2. *On $C_{(m)} \times C_{(n)}$ there exists a unique line bundle $\mathcal{D}_{m,n}$ with the property $\mathcal{D}_{m,n}|_{\{D\}\times C_{(n)}} = \mathcal{L}_D$ and $\mathcal{D}_{m,n}|_{C_{(m)}\times\{E\}} = \mathcal{L}_E$. For $n = m = d$ we denote this bundle by $\mathcal{D}_d$.*

Proof. Let $\varDelta_{i,m+j}$ be the $i, m+j$ diagonal in $C^{\times(m+n)}$ for some $1 \le i \le m,\ 1 \le j \le n$. We take $\mathcal{D}_{m,n}$ to be the line bundle on $C_{(m)} \times C_{(n)}$ which is defined by the (reduced) image of $\varDelta_{i,m+j}$ under the canonical map $q : C^{\times(m+n)} \longrightarrow C_{(m)} \times C_{(n)}$ (this is independent from the choice of i, j). $\mathcal{D}_{m,n}$ satisfies the required properties and it is unique because of the see-saw principle. $\square$

The proof of the following proposition is an easy exercise in intersection of divisors.

Proposition 2.3. *Let $A \in C_{(k)}$ and $B \in C_{(l)}$.*

1. *Let $i : C_{(d)} \longrightarrow C_{(d+k)}$ be the embedding defined by $i(D) = D + A$. Then $i^*\mathcal{L}_M = \mathcal{L}_M$ and $i^*\Delta/2 = \Delta/2 + \mathcal{L}_A$.*
2. *Let $j : C_{(m)} \times C_{(n)} \longrightarrow C_{(m+k)} \times C_{(n+l)}$ be the embedding defined by $j(D, E) = (D+A, E+B)$. Then $j^*\mathcal{D}_{m+k,n+l} = \mathcal{D}_{m,n}+p_1^*\mathcal{L}_B+p_2^*\mathcal{L}_A$, where p_i is the ith projection.*

2.2 Line bundles on the Jacobian of a curve

We denote by $J^d(C)$ the Jacobian of degree d. Let $L \in J^{-d+g-1}(C)$. Then the locus of $\{M \in J^d(C) \ \text{ with } \ h^0(M + L) \geq 1\}$ is of codimension one in $J^d(C)$ and therefore it defines a line bundle on $J^d(C)$ which we denote by θ_L. Let θ denote the principal polarization on $J^d(C)$. We have the following well known propositions.

Proposition 2.4. *Every line bundle on $J^d(C)$ of class $n\,\theta$, for $n \in \mathbb{Z}, n \neq 0$, is of the form $n\,\theta_L$ for some $L \in J^{-d+g-1}(C)$. Moreover, if $L, M \in J^{-d+g-1}(C)$ then we have that $n\,\theta_L = n\,\theta_M$, if and only if, $nL = nM$.*

Remark 2.1. The line bundles of class zero on $J^d(C)$ are parametrized by the elements of $J^0(C)$. For $A \in J^0(C)$, we denote by $T(A)$ the corresponding line bundle on $J^d(C)$ which is defined by $T(A) = \theta_{L_0+A} - \theta_{L_0}$, for some $L_0 \in J^{-d+g-1}(C)$. The definition is independent from the choice of L_0. For $A, B \in J^0(C)$ we have $T(A + B) = T(A) + T(B)$. Also, for $L, M \in J^{-d+g-1}$ we have $T(L - M) = \theta_L - \theta_M$.

Proposition 2.5. *Let $n, m \in \mathbb{Z}$. We choose $M_0 \in J^{m-n}(C)$ and $N_0 \in J^{m+n}(C)$.*

1. *Let $\phi : J^n(C) \longrightarrow J^m(C)$ given by sending A to $A + M_0$. Then $\phi^*\theta_L = \theta_{L+M_0}$.*
2. *Let $\psi : J^n(C) \longrightarrow J^m(C)$ given by sending A to $N_0 - A$. Then $\psi^*\theta_L = \theta_{K-L-N_0}$, where K is the canonical bundle on C.*

Proposition 2.6. *We have $K_{C_{(d)}} = \mathcal{L}_K - \Delta/2$, where K is the canonical bundle of C and $K_{C_{(d)}}$ is the determinant of the cotangent bundle of $C_{(d)}$.*

Proof. This can be deduced from the description of the tangent bundle of $C_{(d)}$, given in [2], Ch. IV, Lemma 2.3, and by applying the GRR-theorem to the projection map $C_{(d)} \times C \longrightarrow C_{(d)}$. □

Let $\mathcal{P}_Q$ be the Poincaré bundle normalized at the point $Q \in C$. For $d > 2g - 2$, the direct image $E = \nu_*\mathcal{P}_Q$, where ν is the projection from $C \times J^d(C)$ to $J^d(C)$, is a vector bundle of rank $d-g+1$. The projectivization

$\mathbb{P}(E)$ of E is a projective bundle over $J^d(C)$ which is isomorphic to the symmetric product $C_{(d)}$. The Abel-Jacobi map $u_d : C_{(d)} \longrightarrow J^d(C)$ is the canonical map of the projective fibration. The tautological line bundle $\mathcal{O}(1)$ of $\mathbb{P}(E)$ turns out to be the line bundle $\mathcal{L}_Q$, see [2], Ch. VII, Prop. 2.1. We have the following Lemma, see [6], Lemma 6.

Lemma 2.1. $\det \nu_* \mathcal{P}_Q = -\theta_{(-d+g-1)Q}$.

Proposition 2.7. *Let $u = u_d : C_{(d)} \longrightarrow J^d(C)$ be the Abel-Jacobi map. Then $u^*\theta_L = \mathcal{L}_{K-L} - \Delta/2$.*

Proof. For $d > 2g-2$ the fibration u has Euler sequence $0 \longrightarrow \mathcal{O} \longrightarrow u^*\nu_*\mathcal{P}_Q \otimes \mathcal{O}(1) \longrightarrow \Omega_u^\vee \longrightarrow 0$. Let ω_u denote the relative dualizing sheaf Ω_u. Then $\omega_u = K_{C_{(d)}}$. By taking determinants, Proposition 2.6 yields that $u^*\theta_{(-d+g-1)Q} = \mathcal{L}_{K+(d-g+1)Q} - \Delta/2$. From the construction of the Poincaré bundle, see for example [6], & 4, we have that $\mathcal{P}_Q|_{\{R\}\times J^d(C)} = T(Q-R)$, for $Q, R \in C$, and therefore $\mathcal{P}_P = \mathcal{P}_Q + \nu^*T(P-Q)$, for $P, Q \in C$. Therefore $\nu_*\mathcal{P}_P = \nu_*\mathcal{P}_Q + T(P-Q)$ and so, see for example [6], Lemma 7, we get that $\mathcal{L}_P = \mathcal{L}_Q + u^*T(Q-P)$. This in turns implies that $\mathcal{L}_L - \mathcal{L}_{(-d+g-1)Q} = u^*T((-d+g-1)Q-L) = u^*\theta_{(-d+g-1)Q} - u^*\theta_L$, see Remark 2.1. Therefore $u^*\theta_L = \mathcal{L}_{K+(d-g+1)Q} - \Delta/2 - \mathcal{L}_L + \mathcal{L}_{(-d+g-1)Q} = \mathcal{L}_{K-L} - \Delta/2$.

For $d \le 2g-2$, we define the maps $i_{P_0} : C_{(d-1)} \longrightarrow C_{(d)}$ with $i_{P_0}(D) = D + P_0$ and $j_{P_0} : J^{d-1}(C) \longrightarrow J^d(C)$ with $j_{P_0}(L) = L + \mathcal{O}(P_0)$. Then, by Propositions 2.5 and 2.3, we have that

$$\begin{aligned} u_{d-1}^*\theta_L &= u_{d-1}^* j_{P_0}^* \theta_{L-\mathcal{O}(P_0)} = i_{P_0}^* u_d^* \theta_{L-\mathcal{O}(P_0)} \\ &= i_{P_0}^*(\mathcal{L}_{K-L+\mathcal{O}(P_0)} - \Delta/2) = \mathcal{L}_{K-L} - \Delta/2 \end{aligned}$$

and the result follows for all d's by inverse induction. $\square$

For $d > 2g-2$, the fibers of the Abel-Jacobi map $u = u_d : C^{(d)} \longrightarrow J^d(C)$ are projective spaces of constant dimension $d-g$. For general d, the symmetric product is isomorphic to the projectivization of a coherent sheaf on $J^d(C)$. Therefore the fibers are, also, projective spaces but their dimension varies. We denote, in any case, by H the hyperplane bundle of a fiber of dim≥ 1. We then have

Proposition 2.8. *Let M be a line bundle on C of degree m. Then the restriction of the line bundle $\mathcal{L}_M$ (resp. $\Delta/2$) to a fiber of the Abel-Jacobi map u_d of dim≥ 1 is equal to $m\,H$ (resp. $(d+g-1)\,H$).*

Proof. For $d > 2g-2$ the first is a consequence of the fact that the class of $\mathcal{L}_M$ in the Neron-Severi group is equal to $m\,x$ and that the tautological bundle of the projective bundle is $\mathcal{L}_Q$ which has class x. The second is then a consequence of Proposition 2.7. To prove the proposition for $d \le 2g-2$, we use inverse induction as in the proof of Proposition 2.7. $\square$

2.3 Line bundles on a family of symmetric products of curves

Let $\pi : \mathcal{C} \longrightarrow B$ be a flat family of smooth curves over a smooth base B. To that we can associate the family $\mathcal{C}^{\times d}$ of the fibered ordinary product and the family $\mathcal{C}_{(d)}$ of the fibered symmetric product. We have the diagram:

$$\begin{tikzcd} \mathcal{C}^{\times d} \arrow[rr, "q"] \arrow[d, "\pi_i"'] \arrow[drr, "\chi"] & & \mathcal{C}_{(d)} \arrow[d, "\phi"] \\ \mathcal{C} \arrow[rr, "\pi"] & & B \end{tikzcd} \tag{2.1}$$

where the maps q, χ, ϕ, π are the canonical ones and π_i is the i-th projection.

We denote by ω the relative dualizing sheaf of the family $\pi : \mathcal{C} \longrightarrow B$. We denote by λ the determinant of the Hodge bundle i.e the line bundle defined as $\lambda = \det \pi_* \omega$. We keep the same notation λ for the pull back $\phi^* \lambda$ (resp. $\pi^* \lambda$) of λ to $\mathcal{C}_{(d)}$ (resp. $\mathcal{C}$). On the family $\mathcal{C}_{(d)}$ we can define, in a similar way as in the case of a fixed curve, the diagonal divisor (or line bundle) which we denote, again, by Δ. On the same lines, we have on $\mathcal{C}_{(d)}$ a line bundle with square equal to Δ, which we denote by $\Delta/2$. We describe now the analogue of $\mathcal{L}_K$, where K is the canonical bundle, for the family $\mathcal{C}_{(d)}$ of symmetric products of curves. Take on $\mathcal{C}^{\times d}$ the line bundle $\sum_{i=1}^{d} \pi_i^* \omega$. This is invariant under the action of the symmetric group and there is therefore a line bundle $\mathcal{L}_\omega$ on $\mathcal{C}_{(d)}$ with the characteristic property $q^* \mathcal{L}_\omega = \sum_{i=1}^{d} \pi_i^* \omega$. This will be the anologue of $\mathcal{L}_K$. Finally, on the family $\mathcal{C}_{(m)} \times_B \mathcal{C}_{(n)}$, we can construct, in the same way as in the case of the fixed curve, the analogue of the line bundle $\mathcal{D}_{m,n}$ (or $\mathcal{D}_d$) which we denote by the same notation.

Proposition 2.9. *The relative dualizing sheaf of the family $\phi : \mathcal{C}_{(d)} \longrightarrow B$ is given by $\omega_\phi = \mathcal{L}_\omega - \Delta/2$.*

Proof. From Proposition 2.6 and the see-saw principle we have that $\omega_\phi = \mathcal{L}_\omega - \Delta/2 + \phi^* \mathcal{B}$, for some line bundle $\mathcal{B}$ on B. We show that $\mathcal{B}$ is the trivial bundle. The map q of Diagram 2.1 is simply ramified along the diagonals. Therefore, $q^* \omega_\phi = \omega_\chi - \sum_{1 \le i < j \le d} \Delta_{ij}$. On the other hand, since $\mathcal{C}^{\times d}$ is a fibered product over B, we have that $\omega_\chi = \sum_{i=1}^{d} \pi_i^* \omega = q^* \mathcal{L}_\omega$. Since now $q^* \Delta/2 = \sum_{1 \le i < j \le d} \Delta_{ij}$, we get that $\chi^* \mathcal{B}$ is trivial and therefore $\mathcal{B}$ is trivial. $\square$

2.4 The Morita line bundle on a family of Jacobians

To the family of curves $\pi : \mathcal{C} \longrightarrow B$ we can associate the family $\psi : \mathcal{J}^d \longrightarrow B$ of Jacobians of degree d. The fiber of ψ over a point $b \in B$ is the Jacobian

$J^d(C_b)$, where $C_b := \pi^{-1}(b)$. The Jacobian fibration $\mathcal{J}^0 \longrightarrow B$ of degree 0 admits a zero section. Let λ be the determinant of the Hodge bundle on B. We denote the line bundle $\psi^*\lambda$ simply by λ.

On $\mathcal{J}^0$ there is a closed 2-form η_0 whose restriction to each fiber is a representative of the class of the principal polarization and whose restriction to the zero section is trivial. The cohomology class of this form takes values in the rationals but the class of the form $\eta = 2\eta_0$ takes values in the integers, see [8]. To prove this, we show that the class $[\eta]$ is the chern class of a line bundle on $\mathcal{J}^0$ which can be constructed as follows: Let X be an abelian variety and L a line bundle on X. We denote by $X^\vee$ the dual abelian variety and let $\phi_L : X \longrightarrow X^\vee$ be the map which sends $x \in X$ to $T_x^*L - L$. The map ϕ_L depends only on the class of L, see [7], ch. 2, (4.6). By using the class of η_0 we can define, in a similar way, a map $\phi_{\eta_0} : \mathcal{J}^0 \longrightarrow \mathcal{J}^{0\vee}$. Let $\Phi = 1 \times \phi_{\eta_0} : \mathcal{J}^0 \longrightarrow \mathcal{J}^0 \times_B \mathcal{J}^{0\vee}$. Let $\mathcal{P}$ denote the Poincaré bundle on $\mathcal{J}^0 \times_B \mathcal{J}^{0\vee}$. By standard theory, see [7], ch. 2, & 5, we can see that $\mathcal{L} := \Phi^*\mathcal{P}$ is a line bundle on $\mathcal{J}^0$ with chern class $[\eta]$. The restriction of $\mathcal{L}$ to a fiber is $2\theta_A$, where A is a line bundle with $2A = K$, and its restriction to the zero section is trivial.

We are giving now another construction of the above line bundle which is more convenient for our purposes. We will work on the Jacobian $\mathcal{J}^{m(2g-2)}, m \in \mathbb{N}, m \geq 1$, which is isomorphic to $\mathcal{J}^0$ under the map which sends L to $L + mK$. Let $u : \mathcal{C}_{(m(2g-2))} \longrightarrow \mathcal{J}^{m(2g-2)}$ be the Abel-Jacobi map. On $\mathcal{C}_{(m(2g-2))}$ we take the line bundle $(2m+1)\mathcal{L}_\omega - \Delta$. From Proposition 2.8, the restriction of this line bundle to the fibers of the map u is trivial. There exists therefore a line bundle $\mathcal{L}_m^{'}$ on $\mathcal{J}^{m(2g-2)}$ with the property $u^*\mathcal{L}_m^{'} = (2m+1)\mathcal{L}_\omega - \Delta$. From Proposition 2.7 the restriction of $\mathcal{L}_m^{'}$ to the fibers of ψ is equal to $2\,\theta_A$, where A is a line bundle with $2A = -(2m-1)K$. We multiply now $\mathcal{L}_m^{'}$ by an appropriate multiple of the pull back of a line bundle on the base S, in order the restriction to the "canonical" section of the fibration $\mathcal{J}^{m(2g-2)} \longrightarrow B$ to become trivial. We denote this line bundle by $\mathcal{L}_m$. Using the isomorphism of $\mathcal{J}^{2g-2}$ with $\mathcal{J}^0$ we get a line bundle, $\mathcal{L}_0$, on $\mathcal{J}^0$. From Proposition 2.5, the restriction of the line bundle $\mathcal{L}_0$ to a fiber of the map ψ is $2\theta_A$, where A is a line bundle with $2A = K$, and its restriction to the zero section is trivial. Hence $\mathcal{L}_0$ is isomorphic to the above constructed line bundle $\mathcal{L}$. We refer to the line bundles $\mathcal{L}_m$, $m \geq 0$, as the "Morita line bundles". To summarize:

Proposition 2.10. *Keeping the above notation, we have that the restriction of the Morita line bundle $\mathcal{L}_m$, $m \geq 0$, to a fiber of $\mathcal{J}^{m(2g-2)}$ is equal to $2\,\theta_A$, where A is a line bundle with $2A = -(2m-1)K$. In particular, the restriction of $\mathcal{L} = \mathcal{L}_0$ to a fiber of $\mathcal{J}^0$ is a totally symmetric line bundle.*

3 Morita's theorems

3.1 The first Theorem

Let $\pi : \mathcal{C} \longrightarrow B$ be a flat family of smooth curves over a smooth base B which admits a section $s : B \longrightarrow \mathcal{C}$. The image $s(B)$ of B defines a divisor on $\mathcal{C}$ which we denote by S. We define the map $i : \mathcal{C} \longrightarrow \mathcal{J}^0$ by sending $P \in C$ to $\mathcal{O}(P) - \mathcal{O}(s(\pi(P))) \in \mathcal{J}^0$. Morita's first theorem, see [8] Theorem 1.3, expressed in terms of line bundles states the following:

Theorem 3.1. *Keeping the above notation, we have that*

$$i^*\mathcal{L} = 2\,\mathcal{O}(S) + \omega + \pi^* s^*(\omega).$$

We are actually going to prove the following more general proposition.

Proposition 3.1. *Let $\delta : \mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)} \longrightarrow \mathcal{J}^0$ be the map which sends $(D, E) \in C_{(d)} \times C_{(d)}$ to $\mathcal{O}(D - E) \in J^0(C)$. Let p_i be the ith projection from $\mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)}$ to $\mathcal{C}_{(d)}$. Then we have*

$$\delta^*\mathcal{L} = 2\,\mathcal{D}_d + p_1^*(\mathcal{L}_\omega - \Delta) + p_2^*(\mathcal{L}_\omega - \Delta).$$

Remark 3.1. Theorem 3.1 corresponds to the special case $d = 1$, see also [3], Theorem 4. Indeed, consider the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[d, "\pi"'] & \mathcal{C} \times_B \mathcal{C} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{C} \\ B \arrow[r, "s"] & \mathcal{C} & \end{tikzcd} \tag{3.1}$$

where $j(x) = (x, s(\pi(x)))$ and p_i is the ith projection. Then $j^*\mathcal{D}_1 = \mathcal{O}(S)$, $j^*\omega_{p_1} = \pi^* s^*\omega$ and $j^*\omega_{p_2} = \omega$.

Proof. We start with a Lemma:

Lemma 3.1. *Let $u : C_{(d)} \times C_{(d)} \longrightarrow J^d(C) \times J^d(C)$ be the map which sends (D, E) to $(\mathcal{O}(D), \mathcal{O}(E))$ and $j : J^d(C) \times J^d(C) \longrightarrow J^0(C)$ be the map which sends (L, M) to $L - M$. Let $\delta_C = j \circ u$. Then $\delta_C^*\,\theta_A = \mathcal{D}_d + p_1^*(\mathcal{L}_{K-A} - \Delta/2) + p_2^*(\mathcal{L}_A - \Delta/2)$.*

Proof. We first determine the $\delta_C^*\,\theta_A|_{\{D\}\times C_{(d)}}$ and $\delta_C^*\,\theta_A|_{C_{(d)}\times\{E\}}$. From Proposition 2.5 we have $j^*\theta_A|_{\{L\}\times J^d(C)} = \theta_{K-A-L}$. Then, by Proposition 2.7, we get $u^*\theta_{K-A-L} = \mathcal{L}_{A+L} - \Delta/2$. Therefore $\delta_C^*\,\theta_A|_{\{D\}\times C_{(d)}} = \mathcal{L}_{A+\mathcal{O}(D)} - \Delta/2$. Similarly, we have $\delta_C^*\,\theta_A|_{C_{(d)}\times\{E\}} = \mathcal{L}_{K-A+\mathcal{O}(E)} - \Delta/2$. The result now follows from the characteristic property of $\mathcal{D}_d$. $\square$

To continue now with the proof of the proposition, we apply Lemma 3.1 to $2\,\theta_A$, where $2A = K$. We get that $2\,\delta_C^*\,\theta_A = 2\mathcal{D}_d + p_1^*(\mathcal{L}_K - \Delta) + p_2^*(\mathcal{L}_K - \Delta)$. We define on $\mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)}$ the line bundle $\mathcal{A} := 2\,\mathcal{D}_d + p_1^*(\mathcal{L}_\omega - \Delta) + p_2^*(\mathcal{L}_\omega - \Delta)$. Then, by Proposition 2.10, the restrictions of $\delta^*\mathcal{L}$ and $\mathcal{A}$ to the fibers of the map $\psi : \mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)} \longrightarrow B$ are isomorphic. Therefore, we have on $\mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)}$ that $\delta^*\mathcal{L} = \mathcal{A} + \psi^*\mathcal{B}$, where $\mathcal{B}$ is a line bundle on B. We show now that $\mathcal{B} = \mathcal{O}$. Let $t : \mathcal{C}_{(d)} \longrightarrow \mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)}$ be the diagonal map which sends $D \in C_{(d)}$ to $(D, D) \in C_{(d)} \times C_{(d)}$. Observe now that $\delta \circ t = \mu$, where $\mu : B \longrightarrow \mathcal{J}^0$ is the zero section. Therefore $t^*\delta^*\mathcal{L} = \mathcal{O}$. To show that $\mathcal{B}$ is trivial, it suffices to show that $t^*\mathcal{A}$ is trivial. We have $p_i \circ t = \mathrm{id}$, and so, $t^*(p_1^*(\mathcal{L}_\omega - \Delta) + p_2^*(\mathcal{L}_\omega - \Delta)) = 2(\mathcal{L}_\omega - \Delta)$. On the other hand we claim that $t^*(\mathcal{D}_d) = -\mathcal{L}_\omega + \Delta$. To prove this we use the commutative diagram:

$$\begin{tikzcd}
\mathcal{C}^{\times d} \arrow[r, "\gamma"] \arrow[d, "q"'] & \mathcal{C}^{\times 2d} \arrow[d, "p"] \\
\mathcal{C}_{(d)} \arrow[r, "t"] & \mathcal{C}_{(d)} \times_B \mathcal{C}_{(d)}
\end{tikzcd} \tag{3.2}$$

where γ is the diagonal map. We have $p^*\mathcal{D}_d = \sum_{1\le i,j\le d} \Delta_{i,d+j}$. Now $\gamma^*\Delta_{i,d+j} = \Delta_{ij}$, if $i \neq j$ and $\gamma^*\Delta_{i,d+i} = -\pi_i^*\omega$, where $\pi_i : \mathcal{C}^{\times d} \longrightarrow \mathcal{C}$ is the ith projection. Therefore, $\gamma^*p^*\mathcal{D}_d = 2\sum_{1\le i<j\le d} \Delta_{ij} - \sum_{i=1}^d \pi_i^*\omega$. Now $q_*\Delta_{ij} = (d-2)!\,\Delta$ and $q_*\pi_i^*\omega = (d-1)!\,\mathcal{L}_\omega$. Therefore, $q_*\gamma^*p^*\mathcal{D}_d = d!\,(-\mathcal{L}_\omega + \Delta)$. Since q is a map of degree $d!$, this proves the claim. □

3.2 The second Theorem

Let $\pi : \mathcal{C} \longrightarrow B$ be a flat family of smooth curves of genus $g \geq 3$ over a smooth base B. We define the map $t : \mathcal{C} \longrightarrow \mathcal{J}^0$ by sending the point $P \in C$ to $(2g-2)\mathcal{O}(P) - K \in J^0(C)$, where K is the canonical bundle on C. Morita's second theorem, see [9] Theorem 1.7 and [3] Theorem 1, expressed in terms of line bundles is the following.

Theorem 3.2. *Keeping the above notation, we have that*

$$t^*\mathcal{L} = 4g(g-1)\,\omega - 12\,\lambda.$$

Proof. We start by recalling the notion of a line bundle on the moduli functor $\mathfrak{M}_{g,1}$ associated to families of smooth curves of genus g with a section. By a line bundle L on $\mathfrak{M}_{g,1}$ we mean the datum of a line bundle L_S on S, for each family $p : \mathcal{X} \longrightarrow S$ of smooth curves with a section $\sigma : S \longrightarrow \mathcal{X}$. These line bundles L_S are moreover required to satisfy some functorial properties with respect to maps between families of curves. There are two natural choices, L and $L^{'}$, of line bundles on $\mathfrak{M}_{g,1}$ which are defined by setting $L_S := \sigma^* \omega_p$ and $L^{'}_S := \det p_* \omega_p$. It turns out that the (integral) Picard group of $\mathfrak{M}_{g,1}$, $g \geq 3$, is freely generated by the L and $L^{'}$, see [1]. To the family $\pi : \mathcal{C} \longrightarrow B$, as in the statement of the theorem, we associate the family $p_1 : \mathcal{X} := \mathcal{C} \times_B \mathcal{C} \longrightarrow \mathcal{C}$ with the diagonal section. Then the "restrictions" of L and $L^{'}$ to the above family are equal to the line bundles ω and λ of the theorem. On the other hand, the line bundle $t^*\mathcal{L}$ is also the "restriction" of a line bundle on the moduli functor. Indeed, to the family $p : \mathcal{X} \longrightarrow S$ with the section σ we associate the corresponding Jacobian fibration $\psi : \mathcal{J}^0_{\mathcal{X}} \longrightarrow S$ and let $\alpha : S \longrightarrow \mathcal{J}^0_{\mathcal{X}}$ be the map defined by $\alpha(s) := (2g-2)\mathcal{O}(\sigma(s)) - K_s$, where K_s is the canonical line bundle of the fiber over s. Then the required bundle is the $\alpha^*\mathcal{L}$. We therefore have that the $t^*\mathcal{L}$ is a linear integral combination of the line bundles ω and λ. To determine the coefficients, it suffices to do that for a specific family of curves having the property that the line bundles ω and λ are $\mathbb{Z}$-linearly independent. We make such a choice by letting $\pi : \mathcal{C} \longrightarrow B$ to be a nondegenerate family of smooth curves of genus $g \geq 3$ over a base B which is a projective smooth curve and we fix this choice till to the end of the proof. That the family is nondegenerate means that the map $\beta : B \longrightarrow \mathcal{M}_g$ from B to the moduli space $\mathcal{M}_g$ of smooth curves of genus g is finite. Such a family exists due to the fact that, for $g \geq 3$, the moduli space $\mathcal{M}_g$ contains one dimensional complete subvarieties. The family then is constructed by making a base change (in order to have a relative curve) and resolution of singularities. We recall that the (rational) line bundle λ is ample on $\mathcal{M}_g$ and hence that its pullback $\lambda = \beta^*\lambda$ to B is ample, since β is a finite map. It is easy now to show that the line bundles ω and λ are $\mathbb{Z}$-linearly independent. Note that we may assume that the generic curve of the family is non-hyperelliptic (for $g \geq 4$ we may assume that all the fibers are non-hyperelliptic).

We use the following construction. Let $E := \pi_*(\omega)$. We then have a map $i : \mathcal{C} \longrightarrow \mathbb{P}(E^\vee)$ which is generically an embedding (it is an embedding away from the locus of hyperelliptic curves). Let $p : \mathbb{P}(E^\vee) \longrightarrow B$ be the projection. The bundle λ is ample on B hence $\mathcal{O}(1) + k\, p^*\lambda$ is a very ample line bundle on $\mathbb{P}(E^\vee)$, for $k \in \mathbb{N}$ big enough, see [5], Ch. II, Proposition 7.10. Therefore we have a map from $\mathcal{C}$ to some projective space $\mathbb{P}^N$, the hyperplane sections of which cut out canonical divisors on the fibers of π. By a genericity argument, we may choose a hyperplane section $\mathcal{N}$ on $\mathcal{C}$ which defines a simply ramified covering of B of degree $2(g-1)$. We have that $\mathcal{O}(\mathcal{N}) = \omega + k\,\lambda$ on $\mathcal{C}$. Let $\mathcal{N}_C$ denote the restriction of the divisor $\mathcal{N}$ to the fiber $C = \pi^{-1}(b)$ over the point b of B. We can then define a divisor $X_{\mathcal{N}}$ on $\mathcal{C}_{(d)}$ which restricts to the divisor $X_{\mathcal{N}_C}$ on the fiber $C_{(d)}$ over b. Let $\mathcal{L}_{\mathcal{N}}$ be the corresponding line bundle. It is easy to check that $\mathcal{L}_\omega = \mathcal{L}_{\mathcal{N}} - kd\,\lambda$.

We define a section ν of the map $\phi := \psi \circ u : \mathcal{C}_{(2g-2)} \longrightarrow \mathcal{J}^{2g-2} \longrightarrow B$ by sending the point b to the point of the fiber $C_{(d)}$ which corresponds to the divisor $\mathcal{N}_C$ on C. We "shift" now to the Jacobian of degree $2g-2$. We denote, again, by t the map $t : \mathcal{C} \longrightarrow \mathcal{J}^{2g-2}$ which sends $P \in C$ to $(2g-2)\mathcal{O}(P) \in J^{2g-2}(C)$ and let $\mathcal{L}_1$ be the Morita bundle on $\mathcal{J}^{2g-2}$. The line bundle $3\mathcal{L}_\omega - \Delta$ is the pull back of a line bundle on $\mathcal{J}^{2g-2}$ which has the same restriction to the fibers of ψ with $\mathcal{L}_1$. Therefore $u^*\mathcal{L}_1 = 3\mathcal{L}_\omega - \Delta - \phi^*\mathcal{B}$, where $\mathcal{B}$ is a line bundle on B. In order to determine $\mathcal{B}$, we find the restriction of $3\mathcal{L}_\omega - \Delta$ to $\Sigma := \nu(B)$.

Lemma 3.2. *We have: $\mathcal{L}_\omega|_\Sigma = 12\lambda|_\Sigma + k(2g-2)\lambda|_\Sigma$ and $\Delta|_\Sigma = 24\lambda|_\Sigma + 3k(2g-2)\lambda|_\Sigma$, where k is the number which appears in the definition of $\mathcal{N}$.*

Proof. We work in the Chow ring. We claim that $\mathcal{L}_\mathcal{N}|_\Sigma$ is equal to $\phi^*\pi_*\mathcal{N}^2|_\Sigma$ and that $\Delta|_\Sigma$ is equal to $\phi^*\pi_*R|_\Sigma$, where R is the ramification divisor of the covering $\mathcal{N}$. Then the lemma follows in the following way: We have that $\mathcal{O}(\mathcal{N}) = \omega + k\,\lambda$. Therefore $\mathcal{N}^2 = \omega^2 + 2k\,\omega\lambda$ and so, $\pi_*\mathcal{N}^2 = 12\,\lambda + 2k(2g-2)\,\lambda$ (we use the well known identity $\pi_*\omega^2 = 12\,\lambda$). Using now that $\mathcal{L}_\omega = \mathcal{L}_\mathcal{N} - k(2g-2)\,\lambda$, we get $\mathcal{L}_\omega|_\Sigma = 12\,\lambda|_\Sigma + k(2g-2)\,\lambda|_\Sigma$. By the adjunction formula we have that $R = \omega + \mathcal{N}|_\mathcal{N}$ and so, $\pi_*R = \pi_*(\omega(\omega + k\,\lambda)) + \pi_*\mathcal{N}^2 = 12\,\lambda + k(2g-2)\,\lambda + 12\,\lambda + 2k(2g-2)\,\lambda = 24\,\lambda + 3k(2g-2)\,\lambda$. Therefore $\Delta|_\Sigma = 24\,\lambda|_\Sigma + 3k(2g-2)\,\lambda|_\Sigma$.

The claim about Δ is set theoretically obvious. We make now a different choice of $\mathcal{N}$, say $\mathcal{N}'$, which intersects $\mathcal{N}$ transversely (we can always do this by genericity). We choose the section $X_{\mathcal{N}'}$ of $\mathcal{L}_\mathcal{N}$ corresponding to the divisor $\mathcal{N}'$. Then, also, the claim about $\mathcal{L}_\mathcal{N}$ becomes set theoretically obvious. It remains to show the transversality of intersections. For this, we introduce local coordinates.

We write $(z;t)$ for the local coordinates of the family of curves $\pi : \mathcal{C} \longrightarrow B$ in a neighbourhood U of some point of the fiber, where t is the local coordinate on B. Let $P_1, \dots, P_d$ be points on a fiber of the map π. For each $i = 1, \dots, d$, we choose a local neighbourhood U_i, with local coordinates $(z_i;t)$ centered at P_i. The ordinary fibered product $\mathcal{C}^{\times d}$ has local coordinates $(z_1, \dots, z_d;t)$ in the neighbourhood $U_1 \times \cdots \times U_d$ centered at $(P_1, \dots, P_d)$. We denote by $q : \mathcal{C}^{\times d} \longrightarrow \mathcal{C}_{(d)}$ the canonical map. The symmetric product has local coordinates $(y_1, \dots, y_d;t)$, where $y_i = \sigma_i(z_1, \dots, z_d)$ is the i-th symmetric polynomial, in the neighbourhood $q(U_1 \times \cdots \times U_d)$, centered at the point $P_1 + \cdots + P_d$. If the P_i's are all different to each other, then the map $q : U_1 \times \cdots \times U_d \longrightarrow q(U_1 \times \cdots \times U_d)$ is an isomorphism and we can take as coordinates in the symmetric product the $(z_1, \dots, z_d;t)$. On the other hand, if $P_1 = P_2$ and the rest are different to each other, then there is an isomorphism of $U_{(2)} \times U_3 \times \cdots \times U_d$ with $q(U_1 \times \cdots \times U_d)$, where

$U = U_1 = U_2$. Now, on $U_{(2)}$ we have local coordinates $(w_1, w_2; t)$, where $w_1 = z_1 + z_2$, $w_2 = z_1 z_2$. Therefore on $q(U_1 \times \cdots \times U_d)$ we have local coordinates $(w_1, w_2, z_3, \ldots z_d; t)$ centered at $P_1 + \cdots + P_d$, with $P_1 = P_2$.

Let $d = 2g - 2$. Let $\mathcal{N}'$ be a covering of B meeting the covering $\mathcal{N}$ transversely at the point P_1 on a fiber over a point $b \in B$ away from their branch loci. We can assume that the local equation for the covering $\mathcal{N}$ of B at the point P_1 is given by $z_1 = 0$ and that of $\mathcal{N}'$ by $z_1 = t$. The section Σ intersects the fiber over b of the map ϕ at a point $P_1 + \cdots + P_d$, with the P_i's different to each other. In terms of the local coordinates $(z_1, \ldots, z_d; t)$ of this point, the local equations of Σ are given, by $z_1 = z_2 = \cdots = z_d = 0$ and the local equation of the divisor $X_{\mathcal{N}'}$ is given by $z_1 = t$. Therefore $X_{\mathcal{N}'}$ intersects transversely the section Σ. On the other hand, at a ramification point the local equation of the covering $\mathcal{N}$ of B is given by $z^2 = t$. Over a point $b \in B$ of the branch locus of the covering $\mathcal{N}$, the diagonal Δ intersects Σ at a point $P_1 + \cdots + P_d$, with $P_1 = P_2 \in \mathcal{N}_C$ and the rest are different to each other. In terms of the above local coordinates $(w_1, w_2, z_3, \ldots, z_d; t)$, the equations of Σ are given by $w_1 = 0$, $w_2 = -t$, $z_i = 0$, $i = 3, \ldots, d$. The equation of the diagonal in the symmetric product is given by $4w_2 = w_1^2$. It is easy then to see that the diagonal intersects the section Σ transversely at the origin: the tangent vector $< 0, -1, 0 \ldots, 0; 1 >$ to the section at the origin does not belong to the tangent space of the diagonal - the latter has normal vector $< 0, 1, 0, \ldots, 0; 0 >$. $\square$

We continue with the proof of the Theorem. We have the diagram:

$$\begin{tikzcd} & \mathcal{C}^{\times(2g-2)} \arrow[d, "q"] & \\ \mathcal{C} \arrow[ur, "j"] \arrow[r, "i"] & \mathcal{C}_{(2g-2)} \arrow[r, "u"] & \mathcal{J}^{2g-2} \end{tikzcd} \tag{3.3}$$

where the map $j : \mathcal{C} \longrightarrow \mathcal{C}^{\times(2g-2)}$ is defined by sending the point $P \in C$ to the point $(p, \ldots, p) \in C^{\times(2g-2)}$. The restriction of $3\mathcal{L}_\omega - \Delta$ to $\Sigma = \nu(B)$ is equal to $12\,\lambda|_\Sigma$, see Lemma 3.2. We have $q^*\mathcal{L}_\omega = \sum_{i=1}^{2g-2} F_\omega^i$ and $q^*\Delta/2 = \sum_{1 \le i < j \le 2g-2} \Delta_{ij}$. It is easy to see that $j^*\Delta_{ij} = -\omega$ and $j^*F_\omega^i = \omega$. We get that $i^*\Delta/2 = -\frac{(2g-2)(2g-3)}{2}\,\omega$ and $i^*\mathcal{L}_\omega = (2g-2)\,\omega$. Therefore $i^*(3\mathcal{L}_\omega - \Delta) = 3(2g-2)\,\omega + (2g-2)(2g-3)\,\omega = 4g(g-1)\,\omega$. Combining the above we get the result. $\square$

We finish this section by stating the following proposition.

Proposition 3.2. *Let* $s = \gcd(d, 2g-2)$ *and let* $m := \frac{2g-2}{s}$ *and* $n := \frac{d}{s}$. *We define the map* $t_d : \mathcal{C}_{(d)} \longrightarrow \mathcal{J}^0$ *by sending the divisor* $D \in C_{(d)}$ *to the* *line bundle* $m\,\mathcal{O}(D) - n\,K_C$ *(for* $d=1$ *the map* t_1 *is the map* t *of Theorem 3.2). We then have* $t_d^*\mathcal{L} = \frac{4(g-1)}{s^2}\,((d+g-1)\mathcal{L}_\omega - 2(g-1)\Delta/2) - 12\frac{d^2}{s^2}\lambda$.

Proof. We use the diagram:

$$\begin{tikzcd} & \mathcal{C}_{(d)} \times_B \ldots \times_B \mathcal{C}_{(d)} \arrow[d, "q_d"] & \\ \mathcal{C}_{(d)} \arrow[ur, "j_d"] \arrow[r, "i_d"] & \mathcal{C}_{(md)} \arrow[r, "u"] & \mathcal{J}^{md} \end{tikzcd} \tag{3.4}$$

where the top product is taken m times. The map j_d is the diagonal map, the map q_d is the canonical one and the map i_d is the multiplication by m. On $\mathcal{C}_{(d)} \times_B \ldots \times_B \mathcal{C}_{(d)}$, we denote by Δ^i the pull back of the diagonal of the ith component, by Δ_{ij} the ij-diagonal and by $\mathcal{L}^i_\omega$ the pull back of $\mathcal{L}_\omega$ via the ith projection. We then have the formulae $q_d^*\Delta = \sum_{i=1}^m \Delta^i + 2\sum_{1\le i<j\le m}\Delta_{ij}$ and $q_d^*\mathcal{L}_\omega = \sum_{i=1}^m \mathcal{L}^i_\omega$. We, also, have $j_d^*\Delta^i = \Delta,\ j_d^*\Delta_{ij} = -\mathcal{L}_\omega + \Delta$ and $j_d^*\mathcal{L}_\omega = \mathcal{L}_\omega$. Working now exactly as in the last paragraph of the proof of Theorem 3.2 we find the $t_d^*\mathcal{L}$ up to the pullback by a line bundle $\mathcal{B}$ on the base B. To determine the latter, we take the map $i : \mathcal{C} \longrightarrow \mathcal{C}_{(d)}$ which sends p to dp. Then $t_d \circ i = t_1 \circ a$, where a is the multiplication - by d/s - map on the Jacobian. By applying Theorem 3.2, we get that $\mathcal{B} = 12\frac{d^2}{s^2}\lambda$. □

3.3 A result related to Morita's work

We state and prove a result related to Morita's work which can be found in [3], Theorem 3. We keep the notation of the previews section. On the Jacobian fibration $\mathcal{J}^{g-1}$ of degree $g-1$, there is a canonical theta divisor, which is the image of the relative symmetric product of degree $g-1$. We denote this divisor and the corresponding line bundle by Θ. We have the following proposition:

Proposition 3.3. *Let* $\xi : \mathcal{C} \longrightarrow \mathcal{J}^{g-1}$ *be the map which sends* $P \in C$ *to* $(g-1)\mathcal{O}(P) \in J^{g-1}(C)$*. Then* $\xi^*\Theta = \frac{g(g-1)}{2}\,\omega - \lambda$.

Proof. We first show the following lemma:

Lemma 3.3. *Let* $u : \mathcal{C}_{(g-1)} \longrightarrow \mathcal{J}^{g-1}$ *be the Abel-Jacobi map. Then* $u^*\Theta = \mathcal{L}_\omega - \Delta/2 - \lambda$.

Proof. The relative dualizing sheaf ω_ψ of the fibration $\psi : \mathcal{J}^{g-1} \longrightarrow B$ is equal to λ. This is a consequence of the well known fact that the same result is true for the relative Jacobian $\mathcal{J}^0 \longrightarrow B$ of degree zero, and the observation that after a finite base change the above two fibrations become isomorphic. The divisor Θ is defined as the image of $\mathcal{C}_{(g-1)}$ under the map u. This is a generic embedding and the locus where it fails to be an embedding is of big codimension. Therefore $u^*\Theta = K_{\mathcal{C}_{(g-1)}} - u^*K_{\mathcal{J}^{g-1}}$ and so, $u^*\Theta = \omega_\phi - u^*\omega_\psi$, where $\phi : \mathcal{C}_{(g-1)} \longrightarrow B$ and $\psi : \mathcal{J}^{g-1} \longrightarrow B$ are the canonical maps. The result now follows by Proposition 2.9. □

To finish the proof of the proposition, we define the map $i : \mathcal{C} \longrightarrow \mathcal{C}_{(g-1)}$ which sends $P \in C$ to $(g-1)P \in C_{(g-1)}$. A calculation similar to that of the last paragraph of the proof of Theorem 3.2, see Diagram 3.3, yields the result. □

Consider now the moduli space $\mathcal{M}_g[2]$ of curves of genus g with level 2 structure (or - to be more precise - the moduli functor), for details see [3]. On $\mathcal{M}_g[2]$ we can consistently choose a theta characteristic for each curve. Let α_C be such a choice. Then we can define a map $j_\alpha : \mathcal{C}[2] \longrightarrow \mathcal{J}^0[2]$ from the universal level 2 curve to the corresponding relative Jacobian of degree zero, by sending $P \in C$ to $(g-1)\mathcal{O}(P) - \alpha_C$. On the other hand, we have an isomorphism $i_\alpha : \mathcal{J}^0[2] \longrightarrow \mathcal{J}^{g-1}[2]$ which sends $L \in J^0(C)$ to $L + \alpha_C$. We can then define on $\mathcal{J}^0[2]$ the line bundle $\Theta_\alpha := i_\alpha^*\Theta$. We state the following proposition, see [3] Theorem 3, which is a corollary of Proposition 3.3:

Proposition 3.4. *Keeping the above notation we have* $j_\alpha^*\Theta_\alpha = \frac{g(g-1)}{2}\,\omega - \lambda$.

Remark 3.2. Proposition 3.4 is actually equivalent to Theorem 3.2, by using the formula $\mathcal{L} = 2\,\Theta_\alpha - \lambda$ which is consequence of the theta transformation formula, see [3].

4 The Morita line bundle and ampleness

4.1 Ampleness of line bundles on the universal Jacobian

Let $\pi : \mathcal{C} \longrightarrow B$ be a nondegenerate flat family of smooth curves, where B is a smooth projective variety of dimension ≥ 1. Note that the fibers of such a family have genus $g \geq 3$ since the moduli spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ do not contain complete subvarieties of positive dimension. Let $\psi : \mathcal{J}^0 \longrightarrow B$ be the Jacobian fibration of degree 0. On that, we consider the Morita line bundle $\mathcal{L}$ and the pull back λ of the determinant of the Hodge bundle. In this section we describe the ample cone in the $\mathcal{L}, \lambda$-plane (with rational coefficient). In the following we keep the same notation for a line bundle and its class. We start with a lemma.

Lemma 4.1. *Let $\phi : X \longrightarrow Y$ be a map of projective varieties. Let L be an ample line bundle on X and M be an ample line bundle on Y. Then the line bundle $L + \phi^* M$ is ample on X.*

Proof. This is a consequence of the Nakai- Moisheson criterion of ampleness by applying the projection formula to the map ϕ. □

Proposition 4.1. *The ample cone of the Jacobian fibration $\psi : \mathcal{J}^0 \longrightarrow B$ in the (rational) $\mathcal{L}, \lambda$-plane is given by the open first quadrant. In other words, the line bundle $a\mathcal{L} + b\lambda$ is ample if and only if $a > 0$ and $b > 0$.*

Proof. We first note that the line bundle λ is not ample on $\mathcal{J}^0$ since it is a pull back. Also, the line bundle $\mathcal{L}$ is not ample since its restriction to the zero section is trivial. For a line bundle of the form $a\mathcal{L} + b\lambda$, with $a, b > 0$, we define the slope to be the fraction $s = a/b$. The question of ampleness depends on the slope. We first claim that if some slope s_0 supports an ample line bundle then any slope s, with $s \leq s_0$, does the same. Indeed, let $a_0\mathcal{L} + b_0\lambda$ an ample line bundle of slope s_0. Then, since λ is an ample line bundle on B, we have by Lemma 4.1 that the line bundle $a_0\mathcal{L} + (b_0 + n)\lambda$ is ample on $\mathcal{J}^0$, for every $n \geq 0$. It suffices therefore to show the existence of ample line bundles as the slope s goes to infinity. For this we use some facts from the theory of theta functions and modular forms on appropriate level structures. We refer to the book [7], Ch. 8, & 9. We denote by $\mathcal{A}_0 = \mathcal{A}_D(D)_0$ the moduli space of polarized abelian varieties with orthogonal level D-structure, where $D = 2nI, n \in \mathbb{N}$ and I the identity matrix. Let $p : \mathcal{X}_0 \longrightarrow \mathcal{A}_0$ be the corresponding family of abelian varieties. We have maps ϕ_D, see Remark 9.3, Ch. 8 in [7], and $\bar{\psi}_D$, see Lemma 9.2, Ch. 8 in [7], such that the following diagram commutes

$$\begin{tikzcd} \mathcal{X}_0 \arrow[d, "p"'] \arrow[dr, "\phi_D"] & \\ \mathcal{A}_0 \arrow[r, "\bar{\psi}_D"] & \mathbb{P}^N \end{tikzcd} \tag{4.1}$$

The map ϕ_D is defined via the theta functions of level D and characteristic zero and ψ_D is defined via the corresponding theta nulls. By Theorem 10.1, Ch. 8 in [7], the map $\bar{\psi}_D$ is an embedding for $n \geq 2$. The line bundle which defines the embedding is (modulo a torsion factor of order $8g$) equal to $1/2\,\lambda$. Also, the restriction of the map ϕ_D on each fiber is an embedding. We can then define a map $\Phi : \mathcal{X}_0 \longrightarrow \mathbb{P}^N \times \mathbb{P}^N$ by $\Phi(x_a) = (\phi_D(x_a), \bar{\psi}_D(a))$, where x_a is a point of $\mathcal{X}_0$ with $p(x_a) = a$. The map Φ is an embedding. We take now the family $\mathcal{J}^0 \longrightarrow B$ and we make a finite base change to $\mathcal{A}_0$. This is a finite map of projective varieties and therefore it does not “effect” the ampleness. Let $\tilde{\mathcal{J}}^0 \longrightarrow \tilde{B}$ be the resulting family. We denote, again, by $\mathcal{L}$ the pull back of the Morita line bundle to $\tilde{\mathcal{J}}^0$. The line bundle which defines the restriction on $\tilde{\mathcal{J}}^0$ of the map ϕ_D is equal (modulo torsion) to $n\,\mathcal{L}+1/2\,\lambda$. This can be seen by observing that the restriction of the hyperplane section on a fiber is n times that of the restriction of $\mathcal{L}$ since both are line bundles of the same class $2n\,\theta$ and of characteristic zero. On the other hand, the restriction of the hyperplane section on the zero section is $1/2\,\lambda$ while that of $\mathcal{L}$ is trivial. The pull back of the hyperplane section $\pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$ to $\tilde{\mathcal{J}}^0$, by the map Φ, is equal to $n\,\mathcal{L} + \lambda$. By taking now the n's to become larger and larger we get that the line bundles of big slope are ample. □

4.2 Ampleness on a family of curves and its symmetric products

Proposition 4.1 combined with Morita's Theorem 3.2 yields the following result which was proved in [4].

Proposition 4.2. *Let B be a smooth projective variety of dimension ≥ 1 and $\pi : \mathcal{C} \longrightarrow B$ a non-degenerate family of smooth curves. Then $a\omega + b\lambda$ is ample if $a > 0$ and $12a + 4g(g-1)b > 0$.*

Remark 4.1. We do not know if the "edge" line bundle $4g(g-1)\omega - 12\lambda$ is ample.

Proof. The map $t : \mathcal{C} \longrightarrow \mathcal{J}^0$ defined in Sect. 3.2 has finite fibers. Therefore, by Proposition 4.1, $t^*(\mathcal{L} + \epsilon\lambda)$ is ample for $\epsilon > 0$. By Theorem 3.2, this is equal to $4g(g-1)\,\omega - (12-\epsilon)\lambda$ and this yields the result. □

We continue now our discussion by considering the family of the dth symmetric product $\phi : \mathcal{C}_{(d)} \longrightarrow B$ of the above family of curves. We want to describe the ample divisors in the $\mathcal{L}_\omega$, $\Delta/2$, λ-space. We associate to the point (a, b, c) the line bundle $a\,\mathcal{L}_\omega + b\,\Delta/2 + c\,\lambda$.

Proposition 4.3. *Let $A = (d+g-1, -2(g-1), -\frac{3\,d^2}{g-1})$ and $B = (d-1, 2, 0)$. Then the line bundle $a\,\mathcal{L}_\omega + b\,\Delta/2 + c\,\lambda$ is ample on $\mathcal{C}_{(d)}$ if the point (a, b, c) lies in the interior of the convex cone, with vertex at the origin, generated by the points A, B and $(0, 0, 1)$.*

Proof. We start with a lemma.

Lemma 4.2. *Let $\phi : \mathcal{C} \times_B \mathcal{C} \times_B \mathcal{C}_{(d-2)} \longrightarrow \mathcal{J}^0$ be the map defined by $\phi(p, q, D) = \mathcal{O}(p-q)$ and let $p : \mathcal{C} \times_B \mathcal{C} \times_B \mathcal{C}_{(d-2)} \longrightarrow \mathcal{C}_{(d)}$ be the canonical map. Then $p_*\phi^*\mathcal{L} = 2((d-1)\mathcal{L}_\omega + 2\Delta/2)$, where by p_* we denote the push forward of cycles.*

Proof. Let π_{12} be the projection to the first two factors. Then $\phi^*\mathcal{L} = \pi_{12}^*\,\delta^*\mathcal{L}$, where δ is the map of Proposition 3.1 (with $d = 1$). According to the same proposition, this is equal to $2\Delta_{12} + p_1^*\omega + p_2^*\omega$, where p_i is the ith projection and Δ_{12} is the 12-diagonal ,i.e., the bundle $\pi_{12}^*\mathcal{D}_1$. It is easy then to see that $p_*\Delta_{12} = \Delta$ and $p_*p_i^*\omega = (d-1)\mathcal{L}_\omega$. $\square$

Let $q : \mathcal{C}^{\times d} \longrightarrow \mathcal{C}_{(d)}$ be the canonical map. We first claim that the line bundle $\mathcal{L}_\omega$ on $\mathcal{C}_{(d)}$ is ample. Indeed, we have $q^*\mathcal{L}_\omega = \otimes_{i=1}^{d}\pi_i^*\omega$ and ω is ample on $\mathcal{C}$ by Proposition 4.2. We then have that the bundle $p^*\mathcal{L}_\omega$ is ample on $\mathcal{C} \times_B \mathcal{C} \times_B \mathcal{C}_{(d-2)}$. The line bundle $\mathcal{L} + \epsilon^{'}\lambda$ is ample on $\mathcal{J}^0$, for every $\epsilon^{'} > 0$. Therefore, by applying Lemma 4.1 to the map ϕ we get that the bundle $2\Delta_{12} + p_1^*\omega + p_2^*\omega + \epsilon^{'}\lambda + \eta^{'}\mathcal{L}_\omega$ is ample on $\mathcal{C} \times_B \mathcal{C} \times_B \mathcal{C}_{(d-2)}$ for every $\epsilon^{'}, \eta^{'} > 0$. We take the push forward via the map p and we get, by Lemma 4.2, that the bundle $(d-1+\eta)\mathcal{L}_\omega + 2\,\Delta/2 + \epsilon\lambda$ is ample on $\mathcal{C}_{(d)}$, for every $\epsilon, \eta > 0$.

Let $t_d : \mathcal{C}_{(d)} \longrightarrow \mathcal{J}^0$ be the map defined in Proposition 3.2. Then $t_d^*(\mathcal{L} + \epsilon^{'}\lambda) = \frac{4(g-1)}{s^2}\,((d+g-1)\,\mathcal{L}_\omega - 2(g-1)\Delta/2) - (12\frac{d^2}{s^2} - \epsilon^{'})\lambda$, where $s = \gcd(d, 2g-2)$. By applying now Lemma 4.1 to the map t_d, we have that the line bundle $(d+g-1+\eta)\,\mathcal{L}_\omega - 2(g-1)\,\Delta/2 - (\frac{3d^2}{g-1} - \epsilon)\,\lambda$ is ample, for any $\epsilon, \eta > 0$. This proves the proposition. $\square$

Remark 4.2. We know that there are families of curves for which the vertical sides of the above region do not support ample divisors. Indeed, take a family of curves which contains a curve C with a g_d^1. The symmetric product $C_{(d)}$ contains a curve F the points of which correspond to the sum of the points of C on a fiber of the map defined by the g_d^1. Let O denote the origin. Then the boundary which lies over the ray OA cannot support an ample line bundle: Such a line bundle will be of the form $k((d+g-1)\,\mathcal{L}_\omega - 2(g-1)\,\Delta/2) + \mu\lambda$, for $k \geq 0$. The restriction of that to the above fiber $C_{(d)}$ will be $k((d+g-1)\,\mathcal{L}_\omega - 2(g-1)\,\Delta/2)$ and this cannot be ample since it is a pull back from the Jacobian by the map t_d which contracts the curve F of $C_{(d)}$. Also, the boundary which lies over the ray OB can never support an ample line bundle: Indeed, by taking the restriction to a fiber we observe that the intersection of the above line bundle with the small diagonal is zero. On the other hand, we do not know if there exist an ample line bundle which corresponds to a point which lies in the convex region spanned by the rays OA and OB or below that (in the direction of the z-axis).

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