PROOF OF THE ERGODIC THEOREM

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Let

$$\frac{dx_i}{dt} = X_i(x_1, \ldots x_n) \qquad (i = 1, \ldots n)$$

be a system of n differential equations valid on a closed analytic manifold M, possessing an invariant volume integral, and otherwise subject to the same restrictions as in the preceding note, except that the hypothesis of strong transitivity is no longer made.

We propose to establish first that, without this hypothesis, we have

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \tau(P) \tag{1}$$

for all points P of the surface σ save for points of a set of measure 0. In other words, there is a "mean time $\tau(P)$, of crossing" of σ for the general trajectory.

The proof of the "ergodic theorem," that there is a time-probability p that a point P of a general trajectory lies in a given volume v of M, parallels that of the above recurrence theorem, as will be seen.

The important recent work of von Neumann (not yet published) shows only that there is convergence *in the mean*, so that (1) is not proved by him to hold for any point P, and the time-probability is not established in the usual sense for any trajectory. A *direct* proof of von Neumann's results (not yet published) has been obtained by E. Hopf.

Our treatment will be based upon the following *lemma:* If $S_\lambda[S'_\lambda]$ is a measurable set on σ, which is invariant under T, except possibly for a set of measure 0, and if for any point P of this set

$$\lim_{n=\infty} \sup \frac{t_n(P)}{n} \geqq \lambda > 0 \left[\lim_{n=\infty} \inf \frac{t_n(P)}{n} \leqq \lambda > 0 \right] \tag{2}$$

then

$$\int_{S_\lambda} t_n(P)dP \geqq \lambda \int_{S_\lambda} dP \quad [\int_{S'_\lambda} t(P)dP \leqq \lambda \int_{S_\lambda} dP. \tag{3}$$

We consider only the first case, for the proof of the second case is entirely similar. In analogy with the preceding note, define the distinct measurable sets $U_1, U_2, \ldots$ on S_λ so that for P in U_n

$$t_n(P) > n(\lambda - \epsilon) \quad (P \text{ not in } U_1, U_2, \ldots, U_{n-1}) \tag{4}$$

The quantity $\epsilon > 0$ is taken arbitrarily. It is, of course, clear that for every point P of S_λ

$$t_n(P) > n(\lambda - \epsilon)$$

for infinitely many values of n, so that all such points belong to at least one of the sets $U_1, U_2, \ldots$. Now, by the argument of the earlier note, we infer

$$\int_{S_\lambda^k} t(P)dP > (\lambda - \epsilon) \int_{S_\lambda^k} dP$$

where $S_\lambda^k = U_1 + U_2 + \ldots + U_k$. But S_λ^k is, for every value of k, a measurable part of the invariant set S_λ and increases toward a limit $U_1 + U_2 + \ldots$ which contains every point of S_λ. Consequently we obtain by a limiting process

$$\int_{S_\lambda} t(P)dP \geqq (\lambda - \epsilon) \int_{S_\lambda} dP$$

for any $\epsilon > 0$, whence the inequality of the lemma.

The recurrence theorem stated results directly from this lemma.

Consider the measurable invariant set of points P on σ for which

$$t_n(P) \geqq n\lambda \tag{5}$$

for infinitely many values of n (see the preceding note). This is a set S_λ to which the lemma applies. Similarly the set of points P on σ for which

$$t_n(P) < n\lambda \tag{6}$$

for infinitely many values of n is a set S'_λ of the kind specified in the lemma.

The set S_λ diminishes and the set S'_λ increases with σ, and both sets taken together exhaust σ. The measure of the set S_λ must tend toward 0 as λ increases. Otherwise it would tend toward an invariant measurable set of positive measure, S*, for which the inequality of the lemma holds for $\lambda = \Lambda$, an arbitrarily large positive quantity, and we should infer

$$\int_{S^*} t(P)dP \geqq \Lambda \int_{S^*} dP$$

for any Λ, which is absurd. Moreover, when λ tends toward 0, S_λ becomes vacuous, since there is a least time of crossing, λ_0. In a similar way, S'_λ increases with λ from a set of zero measure for $\lambda < \lambda_0$ toward the set σ.

If then S_λ and S'_λ are not essentially complementary parts of σ, one decreasing, the other increasing, they must, for certain values of λ, have a common measurable component S_λ^* of positive measure, also invariant under T.

Consider the set of points belonging to S_λ^* such that

$$t_n(P) > n\mu \qquad (\mu > \lambda)$$

for infinitely many values of n. These form an invariant measurable subset $S_{\lambda\mu}^*$ of S_λ^*, which must be of measure 0 for any such μ. Otherwise the inequalities of the lemma would give us simultaneously

$$\int_{S_{\lambda\mu}^*} t(P)dP \geqq \mu\int_{S_{\lambda\mu}^*} dP, \qquad \int_{S_{\lambda\mu}^*} t(P)dP \leqq \lambda\int_{S_{\lambda\mu}^*} dP,$$

which are mutually contradictory.

Hence we infer that all of the points P of S_λ^* save for a set of measure 0, satisfy the inequality

$$t_n(P) \leqq n\mu$$

for any $\mu > \lambda$ and for $n = n_P$ sufficiently large, that is,

$$\lim_{n=\infty} \sup \frac{t_n(P)}{n} \leqq \lambda.$$

Likewise we infer that for all of the points of S_λ^*, save for a set of measure 0, we have

$$\lim_{n=\infty} \inf \frac{t_n(P)}{n} \geqq \lambda.$$

It follows then that for points P of S_λ^*, with the usual exception,

$$\lim_{n=\infty} \frac{t_n(P)}{n} = \lambda. \tag{7}$$

Two such sets S_λ^* belonging to different λ's are evidently distinct except for a set of measure 0. Hence there can exist only a numerable set $S_{\lambda_i}^*(i = 1, 2, \ldots)$ of such sets since each has a positive measure. Except for these values λ_i of λ, S_λ' and S_λ are complementary parts of σ aside from a set of measure 0.

Choose now any two values of λ, say λ,μ with $\lambda < \mu$, not belonging to this numerable set, and consider the points of S_λ which do not belong to S_μ. These form an invariant measurable set $S_{\lambda,\mu}$, such that for any point P of this set

$$\lambda \leqq \lim_{n=\infty} \sup \frac{t_n(P)}{n} \mu \tag{8}$$

and also

$$\lambda \leqq \lim_{n=\infty} \inf \frac{t_n(P)}{n} \leqq \mu, \tag{9}$$

since $S_{\lambda,\mu}$ is essentially identical with the part of S_μ' not in S_λ. We infer then $t_n(P)/n$ oscillates between λ and μ as n tends toward ∞, for all points P of $S_{\lambda,\sigma}$ except a set of measure 0.

By choosing a set of values such as λ,μ sufficiently near together we infer then that for all of the points of σ except a set of measure 0, the oscillation of $t_n(P)/n$, as n becomes infinite, is less than an arbitrary $\delta > 0$.

Obviously then the stated recurrence theorem is true.

It should also be noted that if t_n/P denotes the time to the nth crossing as time decreases, the same result holds if n tends toward $\pm\infty$, *with the same limit* except for a set of points P of measure 0. This follows at once from the fact that (8) may be written

$$\lambda \leqq \limsup_{n = -\infty} \frac{t_n(P)}{n} \cdot \leqq \mu,$$

where P of $S_{\lambda,\mu}$ is replaced by $T^n(P)$; and (9) may be given a corresponding form.

This theorem of recurrence admits of certain evident extensions. In the first place there is no need to restrict attention to the analytic case. Moreover, instead of a single surface σ, any measurable set σ^*, imbedded in a numerable set of distinct ordinary surface elements with $v\cos\theta > d > 0$, throughout, will serve, in which case $t^*(P)$ denotes the time from P on σ^* to the first later crossing of σ^*.

In order to prove the "ergodic theorem" we observe first that a set σ^* can be found which cuts every trajectory except those corresponding to equilibrium and others of total measure 0. This is possible; for a numerable set of distinct ordinary surface elements $\sigma_1, \sigma_2, \ldots$ with $v\cos\theta > d > 0$ can be found which cut every trajectory not corresponding to equilibrium. If we define σ_k as the limit of

$$\sigma_1 + \sigma_{12} + \sigma_{123} + \ldots + \sigma_{1\ldots k}$$

where σ_{12} denotes the set of points P of σ_2 not on a trajectory cutting σ_1, σ_{123} denotes the set of points of σ_3 not on a trajectory cutting σ_1 or σ_2, etc., it will have the desired properties.

Now let v denote any "measurable" volume in the manifold M, and let $\bar{t}(P)$ denote the interval of time during which the point on the trajectory which issues from P on such a set σ^* lies in v before the point $T(P)$ of σ^* is reached. Thus $\bar{t}(P) \leqq t(P)$ in all cases. In addition, $\bar{t}_n(P)$ satisfies the same functional equation as $t(P)$

$$\bar{t}_n(P) = \bar{t}(T^{n-1}(P)) + \bar{t}_{n-1}(P).$$

Hence the same reasoning as before is applicable to show that, except for a set of points P of measure,

$$\lim_{n = \pm\infty} \frac{\bar{t}_n(P)}{n} = \bar{t}(P),$$

where $\bar{\tau}(P) \leqq \tau(P)$; while at the same time, of course,

$$\lim_{n=\pm\infty} \frac{t_n(P)}{n} = \tau(P) > 0.$$

We conclude that the following "ergodic theorem" holds

For any dynamical system of type (1) *there is a definite "ti probability" p that any moving point, excepting those of a set of measure will lie in a region v; that is,*

$$\lim_{t=\pm\infty} \frac{\bar{t}}{t} = p \leqq 1$$

will exist, where t denotes total elapsed time measured from a fi. point and $\bar{t}$ the elapsed time in v.

For a strongly transitive system p is, of course, the ratio of the volume of v to V.

Evidently the germ of the above argument is contained in the lemma. The abstract character of this lemma is to be observed, for it shows that the theorem above will extend at once to function space under suitable restrictions.

It is obvious that $\tau(P)$ and $\bar{\tau}(P)$ as defined above satisfy functional relations of the following type:

$$\int_0^\lambda \lambda dm(S_\lambda) = \int_{S_\lambda} t(P)dP$$

where the integral on the left is a Stieltjes integral, $m(S_\lambda)$ being the measure of S_λ.

ERRATA

CORRECTION TO "A SET OF AXIOMS FOR DIFFERENTIAL GEOMETRY"

BY O. VEBLEN AND J. H. C. WHITEHEAD

In the last sentence but one of § 9 we stated that the union of all normal coördinate systems at a point Q, for a given coördinate system $P \longrightarrow x$, exists. This is not necessarily true, as may be seen by considering normal coördinate systems for a cylinder with a locally Euclidean metric.

* These PROCEEDINGS, **17**, 551–562 (1931).