SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

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1. Introduction

1.1. **What is this?** This is a first attempt to organize a list of some open problems on three closely related topics:

(1) The limiting behavior of single and multiple ergodic averages.

(2) Single and multiple recurrence properties of measure preserving systems.

(3) Universal patterns, meaning, patterns that can be found in every set of integers with positive upper density, and related problems on higher dimensions.

The list of problems is greatly influenced by my personal interests and is by no means meant to be a comprehensive list of open problems in the area widely known as ergodic Ramsey theory. Almost exclusively, problems related to actions of commuting measure preserving transformations are considered, and even within this confined class there are a few important topics not touched upon (for instance, the richness of the return times in various multiple recurrence results). For material and a list of problems that goes beyond the scope of this set of notes we refer the reader to the survey articles [1, 2, 3] and the references therein.

Whenever appropriate, I include the "original source" and a short history of the problem, as well as a hopefully accurate and up to date list of related work already done. I plan to update these notes from time to time, so you are welcome to contact me and help me improve them.

1.2. **The general theme.** A very general framework that can be used to describe the bulk of the problems listed below is the following: We are given a measure space $(X, \mathcal{X}, \mu)$ with $\mu(X) = 1$, invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$, bounded measurable functions $f_1, \ldots, f_\ell \colon X \to \mathbb{C}$, and sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$. In some cases we also consider sequences that depend on several integer variables, but let's stick to the single variable case for the time being.

The first family of problems concerns the study of the limiting behavior of the so called multiple ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_1(n)} f_1 \cdot \ldots \cdot T_\ell^{a_\ell(n)} f_\ell \tag{1}$$

where $Tf := f \circ T$ and $T^k := T \circ \cdots \circ T$. One would like to know whether these averages converge as $N \to \infty$ (in $L^2(\mu)$ or pointwise), find some structured factors that control their limiting behavior (called characteristic factors), and if possible, find a formula, or a usable way to extract information, for the limiting function. When $\ell = 1$, such problems have been studied extensively

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and in several cases solved (see the survey paper [10] for a variety of related results). Our main concern here is to study the averages (1) when $\ell \geq 2$. To get manageable problems, one typically restricts the class of eligible sequences, usually to be polynomial sequences, sequences arising from smooth functions, sequences related to the prime numbers, or random sequences, and also assumes that the transformations commute, or, to get started, that they are all equal. On the other hand, because of the nature of the implications in combinatorics that we are frequently interested in, it is not desirable to assume anything about the particular structure of each individual measure preserving transformation. Typically, the tools used to attack such problems include (*i*) elementary uniformity estimates, (*ii*) ergodic structure theorems (like the one in [9]), and (*iii*) equidistribution results on nilmanifolds. More on that on subsequent sections.

The second family of problems concerns the study of expressions of the form

$$\mu(A \cap T_1^{-a_1(n)} A \cap \cdots \cap T_\ell^{-a_\ell(n)} A) \tag{2}$$

where $A \in \mathcal{X}$ has positive measure. One wants to know whether such expressions are positive for some $n \in \mathbb{N}$, or even better, for lots of $n \in \mathbb{N}$ (for instance on the average), and if possible, get some explicit lower bound that depends only on the measure of the set A and on ℓ (optimally this is going to be of the form $(\mu(A))^{\ell+1}$). Such multiple recurrence results are typically obtained by carrying out an in depth analysis of the limiting behavior of the averages (1). Usually they are not hard if an explicit formula of the limiting function is known, but they can be very tricky in the absence of such a formula, even when we work with very special systems of algebraic nature.

Concerning the third family of problems, and restricting ourselves to subsets of $\mathbb{Z}$, one is interested to know, for example, whether every set of integers with positive upper density[1] contains patterns of the form

$$m,\ m + a_1(n), \ldots,\ m + a_\ell(n)$$

for some (or lots of) $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. A typical instance is the celebrated theorem of Szemerédi [11] stating that every set of integers with positive upper density contains arbitrarily long arithmetic progressions (this corresponds to the case $a_i(n) = in$ for $i = 1, \ldots, \ell$, $\ell \in \mathbb{N}$). Using a correspondence principle of H. Furstenberg, one can translate such statements to multiple recurrence statements in ergodic theory; an equivalent problem is then to show that the expressions (2) are positive for some $n \in \mathbb{N}$ when all the measure preserving transformations $T_1, \ldots, T_\ell$ are equal. Similar questions can be asked on higher dimensions, concerning patterns that can be found on subsets of $\mathbb{Z}^d$ with positive upper density. Such questions correspond to multiple recurrence statements when the transformations $T_1, \ldots, T_\ell$ commute. This approach was originally used by H. Furstenberg in [5] to give an alternate proof of Szemerédi's theorem using ergodic theory. Subsequently, H. Furstenberg and Y. Katznelson gave the first proof of the multidimensional Szemerédi theorem [6] and the density Hales-Jewett theorem [7], and V. Bergelson and A. Leibman proved the polynomial extension of Szemerédi's theorem [4] (currently no proof that avoids ergodic theory is known for this result). And the story does not end there, in the last two decades new powerful tools in ergodic theory were developed and used, and are currently being used, to prove several other deep results in density Ramsey

[1]The *upper density* $\bar{d}(E)$ of a set $E \subset \mathbb{Z}^d$ is defined by $\bar{d}(E) := \limsup_{N\to\infty} \frac{|E\cap[-N,N]^d|}{|[-N,N]^d|}$.

theory. The reader will find several such applications in subsequent sections and the extended bibliography section. Several additional applications can be found in the survey articles and the references therein [1, 2, 3].

Recently, an additional motivation for studying such problems has surfaced. It has to do with potential implications in number theory, in particular a connection to problems of finding patterns in the set of the prime numbers. Knowing that every set of integers with positive upper density contains patterns of a certain sort could be an important first step towards proving an analogous result for the set of primes. This idea originates from work of B. Green and T. Tao [8], where it was used to show that the primes contain arbitrarily long arithmetic progressions. It was also subsequently used by T. Tao and T. Ziegler [12] to show that the primes contain arbitrarily long polynomial progressions.

1.3. **General conventions and notation.** We are going to use the following notation: $\mathbb{N} := \{1, 2, \ldots\}$, $T^k := T \circ \cdots \circ T$, $Tf := f \circ T$, if $(X, \mathcal{X}, \mu, T)$ is a measure preserving system then $\mathcal{I}_T := \{A \in \mathcal{X} \colon T^{-1}A = A\}$, $\mathcal{K}_{\text{rat}} := \bigvee_{d\in\mathbb{N}} \mathcal{I}_{T^d}$. We use the symbol $\ll$ when some expression is majorized by a constant multiple of some other expression.

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