Some new multiple ergodic theorems and related open problems

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Three related topics

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2. Multiple recurrence properties of measure preserving systems.
3. Analysis of the limiting behavior of multiple ergodic averages.

Three related topics (Model case)

1. If $d(E) > 0$, then for every $\ell \in \mathbb{N}$ there exists $n \in \mathbb{N}$ s.t.

$$d(E \cap (E - n) \cap \cdots \cap (E - \ell n)) > 0.$$

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2. If $(X, \mathcal{X}, \mu, T)$ is a measure preserving system and $A \in \mathcal{X}$ with $\mu(A) > 0$, then there exists $n \in \mathbb{N}$ s.t.

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3. If $f \in L^\infty(\mu)$, $f \geq 0$, and $\int f\, d\mu > 0$, then

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int f \cdot T^n f \cdot \ldots \cdot T^{\ell n} f \, d\mu > 0.$$

Three related topics (a more general case)

Given sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$, determine whether

1. For every $E \subset \mathbb{N}$ with $d(E) > 0$, there exists $n \in \mathbb{N}$ s.t.

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$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int f \cdot T^{a_1(n)} f \cdot \ldots \cdot T^{a_\ell(n)} f \, d\mu > 0.$$

Multiple ergodic averages

- Such problems lead to the study of the limiting behavior (in $L^2(\mu)$) of the following multiple ergodic averages

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- Higher dimensional problems lead to the study of

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_1(n)}f_1 \cdot \ldots \cdot T_\ell^{a_\ell(n)}f_\ell$$

where $T_1, \ldots, T_\ell$ are commuting measure preserving transformations acting on the same probability space.

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- **Best case scenario:** For every ergodic system

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But it does not happen very often...

Three techniques

1. Use the Host-Kra decomposition.
2. Use extensions.
3. Compare with simpler averages.

First technique: Use the Host-Kra decomposition

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Definition (Gowers-Host-Kra seminorms)

Given an ergodic system $(X, \mathcal{X}, \mu, T)$ and $f \in L^\infty(\mu)$ we define

$$|||f|||_1 = \Big| \int f \, d\mu \Big|, \quad |||f|||_{k+1}^{2^{k+1}} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |||\bar{f} \cdot T^n f|||_k^{2^k}.$$

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Examples

$$|||f|||_2^4 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int \bar{f} \cdot T^n f \, d\mu \right|^2,$$

$$|||f|||_3^8 = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int f \cdot T^m \bar{f} \cdot T^n \bar{f} \cdot T^{m+n} f \, d\mu \right|^2.$$

The more seminorms are 0 the more uniformly/randomly distributed f is for our purposes and the easier it is to deal with f.

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Definition (Nilsequences)

A *k-step nilsequence* is a uniform limit of sequences $(\mathcal{N}(n))$ of the form

$$\mathcal{N}(n) = F(b^n\Gamma)$$

where $X = G/\Gamma$ is a k-step nilmanifold, $b \in G$, and $F\colon X \to \mathbb{C}$ is Riemann integrable (some people prefer F to be continuous).

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$$(e^{i[n\alpha]n\beta}), \quad (e^{i([[n^2\alpha]n\beta]n\gamma-[n\delta]^3n\zeta)}).$$

There are various tools available to study the distribution of nilsequences.

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Theorem (Host, Kra (05))

Let $k \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T)$ be an ergodic system, and $f \in L^\infty(\mu)$. Then for every $\varepsilon > 0$ there exist functions $f_{er}, f_{un}, f_{st} \in L^\infty(\mu)$ such that

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Arithmetic variants were proved recently by **Green, Tao, Ziegler** (11) and **Szegedy** (11).

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Suppose we want to show that the averages

$$A_N(f_1, \ldots, f_\ell) = \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell$$

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converge in $L^2(\mu)$. Strategy:

1. Apply van der Corput's lemma to get the seminorm estimates:

$$\|A_N(f_1,\ldots,f_\ell)\|_{L^2(\mu)}\ll\min_{i=1,\ldots,\ell}|\!|\!|f_i|\!|\!|_\ell.$$

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2. Use the Host-Kra decomposition to deduce that

$$A_N(f_1,\ldots,f_\ell) \sim^{L^2(\mu)} A_N(f_{1,st},\ldots,f_{\ell,st}) = \frac{1}{N}\sum_{n=1}^{N} \mathcal{N}_x(n).$$

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3. If $(\mathcal{N}(n))$ is a nilsequence, then $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathcal{N}(n)$ exists.

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where

$$|||f|||^4_{T,S,\mu} = \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f \cdot T^m \bar{f} \cdot S^n \bar{f} \cdot T^m S^n f \, d\mu.$$

Second technique: Use extensions

- Key idea:

Theorem (Host '09)

There exists an **extension** $(\tilde{X}, \tilde{\mu}, \tilde{T}, \tilde{S})$ *of* (X, μ, T, S) *such that*

$$|||\tilde{f}|||_{\tilde{T},\tilde{S},\tilde{\mu}} = 0 \Leftrightarrow \tilde{f} \perp \mathcal{I}_{\tilde{T}} \vee \mathcal{I}_{\tilde{S}}.$$

(In fact $\tilde{X} = X^4$, $\tilde{T} = (id, T, id, T)$, $\tilde{S} = (id, id, S, S)$, *and*
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- If $\tilde{f}_3 \in \mathcal{I}_{\tilde{T}} \vee \mathcal{I}_{\tilde{S}}$, then mean convergence of

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follows from the mean ergodic theorem, so we are done!

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Unfortunately, this approach has not proven as useful for averages with non-linear iterates. An ongoing project by **Austin** may change that.

Third technique: Compare with something easier

Suppose $(a(n))$ enjoys randomness features (eg primes, random sequences) and we want to show that the averages

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Idea: Compare with the un-weighted averages and show that the difference converges to zero.

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Applying van der Corput's lemma twice one expects to get

$$\left\| \frac{1}{N} \sum_{n=1}^{N} (w(n) - 1) \cdot T^n f \cdot S^n g \right\|_{L^2(\mu)} \ll \| w(n) - 1 \|_{U_3(\mathbb{N})}$$

where $\| z(n) \|_{U_3(\mathbb{N})}^8$ is equal to

$$\limsup_{N\to\infty} \frac{1}{N^2} \sum_{1\leq m,n\leq N} \left| \frac{1}{N} \sum_{h=1}^{N} z(h) \cdot \bar{z}(h+m) \cdot \bar{z}(h+n) \cdot z(h+m+n) \right|^2.$$

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Applicable to the primes (**F., Host, Kra** (08) + **Green, Tao** (10)) and to some random sequences of zero density (**F., Lesigne, Wierdl** (11)).

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$$\frac{1}{\pi(N)} \sum_{n \in \mathbb{P} \cap [1,N]} T^n f \cdot S^n g \sim \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) \cdot T^n f \cdot S^n g$$

converge in $L^2(\mu)$, where T, S are commuting mpt.

Third technique: Compare with something easier

An example:

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converge in $L^2(\mu)$, where T, S are commuting mpt.
Main idea: Exploit the randomness of the primes and show that

$$\frac{1}{N} \sum_{n=1}^{N} \Lambda(n) \cdot T^n f \cdot S^n g - \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^n g \to^{L^2(\mu)} 0.$$

Third technique: Compare with something easier

Two applications of van der Corput's inequality give

$$\left\| \frac{1}{N} \sum_{n=1}^{N} (\Lambda(n) - 1) \cdot T^n f \cdot S^n g \right\|_{L^2(\mu)} \ll \|\!|\Lambda(n) - 1|\!\|_{U_3(\mathbb{N})}.$$

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To finish the proof we need a variant of the previous argument and the following deep result from number theory:

Theorem (Green, Tao (10))

If $W = k!$ *and* $\Lambda_k(n) = \frac{\phi(W)}{W}\Lambda(Wn+1)$, *then*

$$\lim_{k\to\infty} \|\Lambda_k(n) - 1\|_{U_3(\mathbb{N})} = 0.$$

Results and problems: Polynomial sequences

Theorem (Host, Kra (05), Leibman (05))

If $p_1, \dots, p_\ell$ are integer polynomials, then the averages

$$\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)}f_1 \cdot \ldots \cdot T^{p_\ell(n)}f_\ell$$

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converge in $L^2(\mu)$.

Key ingredients:

- **Bergelson**-PET to get seminorm estimates.
- The Host-Kra decomposition result.
- Qualitative equidistribution results on nilmanifolds (**Leibman** (05)).

Results and problems: Polynomial sequences

Theorem (Tao (08))

If the mpt $T_1, \ldots, T_\ell$ commute, then the averages

$$\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell$$

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Tao's proof did not use ergodic theory, he worked on a finitary setup.

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Infinitary proof by **Towsner** (09), ergodic proof by **Austin** (10), a variant by **Host** (09).

Key ingredient in the last two proofs: Extensions.

Results and problems: Polynomial sequences

Theorem (Chu, F., Host (11))

If the mpt $T_1, \ldots, T_\ell$ *commute, and* $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ *have* **distinct degrees,** *then the averages*

$$\frac{1}{N} \sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell$$

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converge in $L^2(\mu)$.

Key ingredients:

- PET induction to get seminorm estimates. (**Hardest step.**)
- The Host-Kra decomposition result.
- Qualitative equidistribution on nilmanifolds (**Leibman** (05)).

Results and problems: Polynomial sequences

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If the mpt $T_1, \ldots, T_\ell$ commute and $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$, show that the averages

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For example, the case $\ell = 2$ and $p_1(n) = p_2(n) = n^2$ is open.

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Problem

If $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are **rationally independent** *and have zero constant term, show that for every $\varepsilon > 0$*

$$\mu(A \cap T_1^{-p_1(n)} A \cap \cdots \cap T_\ell^{-p_\ell(n)} A) \geq \mu(A)^{\ell+1} - \varepsilon$$

for some $n \in \mathbb{N}$.

Results and problems: Smooth functions

Theorem (F. (10))

For every $c \geq 0$ *not an integer, and* $\ell \in \mathbb{N}$, *the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{[n^c]}f_1 \cdot T^{2[n^c]}f_2 \cdot \ldots \cdot T^{\ell[n^c]}f_\ell$$

converge $L^2(\mu)$ *and their limit is* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell$.

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Key ingredients:

- PET induction to get seminorm estimates.
- The Host-Kra decomposition result.
- Quantitative equidistribution on nilmanifolds (**Green, Tao** (11)).

The same also holds for Hardy sequences of polynomial growth that stay logarithmically away from polynomials.

Results and problems: Smooth functions

Theorem (F. (10))

If $c_1, \ldots, c_\ell \geq 0$ *are* **distinct non-integers,** *then*

$$\frac{1}{N}\sum_{n=1}^{N} T^{[n^{c_1}]}f_1 \cdot \ldots \cdot T^{[n^{c_\ell}]}f_\ell \to^{L^2(\mu)} \int f_1 \, d\mu \cdot \ldots \cdot \int f_\ell \, d\mu$$

for every **ergodic** *system.*

Corollary

For **every** *system and set* $A \in \mathcal{X}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-[n^{c_1}]}A \cap \cdots \cap T^{-[n^{c_\ell}]}A) \geq (\mu(A))^{\ell+1}.$$

Results and problems: Smooth functions

Problem

Find an explicit sequence $(a(n))$ *that* **grows faster than polynomials** *(i.e.* $\log(a(n))/n \to \infty$*), such that the following averages converge*

$$\frac{1}{N}\sum_{n=1}^{N} T^{a(n)} f_1 \cdot T^{2a(n)} f_2.$$

You can try $a(n) = [n^{\log\log n}]$.

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You can try $a(n) = [n^{\log\log n}]$.

Problem

Show that for every $c \geq 0$ *and commuting mpt* T_1, T_2 *the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{[n^c]}f_1 \cdot T_2^{[n^c]}f_2$$

converge in $L^2(\mu)$*.*

Results and problems: Prime numbers

We denote by $\mathbb{P}$ the set of prime numbers and $\pi(N) = N/\log N$.

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converge in $L^2(\mu)$.

The proof ultimately relies on the Host-Kra decomposition and also uses some number theoretic input by **Green and Tao**:

- The modified von Mangoldt function has a pseudorandom majorant (08).
- The modified von Mangoldt function minus 1 is asymptotically orthogonal to nilsequences (11).

Results and problems: Prime numbers

Theorem (F., Host, Kra (11))

If $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$, $T_1, \ldots, T_\ell$ commuting mpt, then the averages

$$\frac{1}{\pi(N)} \sum_{n \in \mathbb{P} \cap [1,N]} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell$$

converge in $L^2(\mu)$ **conditionally** *to the convergence of the averages $\frac{1}{N} \sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell$.*

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- Compare with deterministic averages and use PET induction to estimate the difference.
- Use the uniformity of the modified von Mangoldt function minus 1 (**Green, Tao, Ziegler** (11)).

Results and problems: Prime numbers

Problem

Show that for every $c \geq 0$ the averages

$$\frac{1}{\pi(N)} \sum_{n \in \mathbb{P} \cap [1,N]} T^{[n^c]} f_1 \cdot T^{2[n^c]} f_2$$

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Results and problems: Random sequences

Form a sequence $(a_n(\omega))$ by picking, independently, an integer $n \in \mathbb{N}$ to be a member of the sequence with probability $\sigma_n \in [0, 1]$.

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Theorem (F., Lesigne, Wierdl (11))

If $\sigma_n = n^{-c}$ with $\mathbf{c} \in (\mathbf{0}, \mathbf{1}/\mathbf{14})$, then ω-almost surely, for all commuting mpt T, S, and $f, g \in L^\infty(\mu)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{a_n(\omega)} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)$$

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where the the convergence is **pointwise.**

- Compare with simpler averages and use van der Corput to estimate the difference.
- Use the randomness of the random variables to show that the difference converges to 0 pointwise.

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Problem

Show that the previous results hold when $\sigma_n = n^{-c}$ where $\mathbf{c} \in (\mathbf{0}, \mathbf{1})$.

Commercial

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Problem

Determine the structure (with no errors!) of the sequence

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Is it true that $(A(n))$ *is a mixture of* 2*-step nilsequences?*

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THANK YOU!