SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

NIKOS FRANTZIKINAKIS

1. Problems related to polynomial sequences

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by polynomial sequences, and related applications to multiple recurrence.

1.1. **Powers of a single transformation.** Let $\mathcal{P} := \{p_1, \ldots, p_\ell\}$ be a family of integer polynomials that are *essentially distinct*, meaning, all polynomials and their differences are nonconstant. First, we consider averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)}f_1 \cdot \ldots \cdot T^{p_\ell(n)}f_\ell, \tag{1}$$

where $(X, \mathcal{X}, \mu, T)$ is an invertible measure preserving system and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. We remark that all the mean convergence results stated in this section work equally well for averages of the form $\frac{1}{\Phi_N}\sum_{n\in\Phi_N}$ in place of the averages $\frac{1}{N}\sum_{n\in N}$ where $(\Phi_N)_{N\in\mathbb{N}}$ is any Følner sequence of subsets of $\mathbb{N}$.

Before discussing some problems related to the characteristic factors of the averages (1), we state a result of B. Host and B. Kra [32] and A. Leibman [35] that gives useful information about their structure.

Theorem. *There exists $d = d(\mathcal{P})$ such that the factor $\mathcal{Z}_{d,T}$ is characteristic for mean convergence of the averages* (1)*.*

We emphasize that the value of $d(\mathcal{P})$ in the previous statement does not depend on the system or the functions involved. Given a family of polynomials $\mathcal{P}$, we denote by $d_{min}(\mathcal{P})$ the minimal value of $d(\mathcal{P})$ that works in the previous theorem. This value is in general hard to pin down and depends on the algebraic relations that the polynomials satisfy. For instance, we know that $d_{min}(\{n, 2n, \ldots, \ell n\}) = \ell - 1$ ([31, 45]), and $d_{min}(\mathcal{P}) = 1$ when $\mathcal{P}$ consists of at least two rationally independent polynomials ([22, 24]). But it is not only linear relations between the polynomials that matter, for instance, we know that $d_{min}(\{n, 2n, n^2\}) = 2$ while $d_{min}(\{n, 2n, n^3\}) = 1$ ([21, 36]). More examples of families $\mathcal{P}$ where $d_{min}(\mathcal{P})$ has been computed can be found in [21, 36]. Furthermore, in [36] a (rather complicated) algorithm is given for computing this value. Despite such progress, the following is still open (the problem is implicit in [11] and was stated explicitly in [36]):

Problem 1. *If $|\mathcal{P}| \geq 2$, show that $d_{min}(\mathcal{P}) \leq |\mathcal{P}| - 1$.*

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The estimate is known when $|\mathcal{P}| = 2, 3$ ([21]) and it is open for $|\mathcal{P}| = 4$. The problem is even open when one is restricted to the class of Weyl systems, meaning, systems of the form $(\mathbb{T}^d, \mathcal{B}_{\mathbb{T}^d}, m_{\mathbb{T}^d}, T)$ where $T\colon \mathbb{T}^d \to \mathbb{T}^d$ is a unipotent affine transformation. We denote with $d^{\mathrm{W}}_{min}(\mathcal{P})$ the minimum value of $d(\mathcal{P})$ such that the factor $\mathcal{Z}_{d(\mathcal{P}),T}$ is characteristic for mean convergence of the averages (1) for all Weyl systems (properties of $d^{\mathrm{W}}_{min}(\mathcal{P})$ were studied in [11]).

Special Case of Problem 1. *If $|\mathcal{P}| \geq 2$, show that $d^{W}_{min}(\mathcal{P}) \leq |\mathcal{P}| - 1$.*

This problem was first stated in [11] (set $W(P) := d^{W}_{min}(\mathcal{P}) + 1$ in the remark after Proposition 5.3). The estimate is known when $|\mathcal{P}| = 2, 3, 4$ ([21, 40]) and it is open when $|\mathcal{P}| = 5$.

Interestingly, no example is known where $d_{min}(\mathcal{P}) \neq d^{\mathrm{W}}_{min}(\mathcal{P})$, so it is natural to suspect that these two values are always equal.

Problem 2. *Show that $d_{min}(\mathcal{P}) = d^{W}_{min}(\mathcal{P})$.*

This problem was first stated in [11]. The identity is known when $|P| = 3$ ([21]) and is open when $|P| = 4$. Obviously one has $d^{\mathrm{W}}_{min}(\mathcal{P}) \leq d_{min}(\mathcal{P})$. Some bounds in the other direction are given in [36].

Mean convergence of the averages (1) was established after a long series of intermediate results; the papers [26, 16, 17, 18, 28, 41, 30, 31, 45] dealt with the important case of linear polynomials, and using the machinery introduced in [31], convergence for arbitrary polynomials was finally obtained by B. Host and B. Kra in [32] except for a few cases that were treated by A. Leibman in [35].

Theorem. *Let $(X, \mathcal{X}, \mu, T)$ be an invertible measure preserving system, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ be functions, and $p_1, \ldots, p_\ell$ be integer polynomials. Then the averages* (1) *converge in the mean as $N \to \infty$.*

Furthermore, explicit formulas for the limit can be given for special families of polynomials [44, 22, 24, 21, 36], but no such formula is known for general families of polynomials.

In most cases, it is still unknown whether mean convergence can be boosted to pointwise convergence. We mention two particular cases that are open:

Problem 3. *Show that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^n x)\cdot f_2(T^{2n}x)\cdot f_3(T^{3n}x), \quad \text{or the averages} \quad \frac{1}{N}\sum_{n=1}^{N} f_1(T^n x)\cdot f_2(T^{n^2}x),$$

converge pointwise.

Pointwise convergence of the averages (1) is known when $\ell = 1$ [12] and is also known when $\ell = 2$ and both polynomials are linear [13] (see also [19] for an alternate proof). In all other cases the problem is open even for weak mixing systems. Partial results that deal with special classes of transformations can be found in [1, 2, 6, 7, 20, 34, 38, 39].

1.2. **Commuting transformations.** Throughout this section $(X, \mathcal{X}, \mu)$ is a probability space, $T_1, \ldots, T_\ell\colon X \to X$ are commuting, invertible measure preserving transformations, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ are functions, and $p_1, \ldots, p_\ell$ are polynomials with integer coefficients.

We start with the following result of V. Bergelson, B. Host, and B. Kra [9]:

Theorem. *For ergodic systems one has the decomposition*

$$\int f_0 \cdot T^n f_1 \cdot \ldots \cdot T^{\ell n} f_\ell \, d\mu = N(n) + e(n)$$

where $(N(n))$ *is an* ℓ*-step nilsequence and* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N |e(n)| = 0$.

A more general result that uses polynomial iterates in place of the linear iterates was proved recently in [37]. A key ingredient in the proof of the previous theorem is the fact that the factor $\mathcal{Z}_{\ell,T}$ is characteristic for convergence of the averages $\frac{1}{N}\sum_{n=1}^N |\int f_0 \cdot T^n f_1 \cdot \ldots \cdot T^{\ell n} f_\ell \, d\mu|$. When one uses commuting transformations in place of powers of the same transformation an analogous property fails, nevertheless, there are no known examples of multicorrelation sequences of commuting transformations that are genuinely different than nilsequences.

Problem 4. *Is it true that one always has the decomposition*

$$\int f_0 \cdot T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell \, d\mu = N(n) + e(n)$$

where $(N(n))$ *is an* ℓ*-step nilsequence and* $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N |e(n)| = 0$*?*

The question is open even when $\ell = 2$. Notice that we make no ergodicity assumptions, so in particular this problem is open even when $\ell = 2$ and $T_2 = T_1^2$.

We move to some problems related to convergence properties of multiple ergodic averages. A very natural problem (stated explicitly in [8] but was advertised long before 1996 by H. Furstenberg and others) is to extend the mean convergence result involving polynomial iterates of a single transformation to several commuting transformations:

Problem 5. *Show that the averages*

$$\frac{1}{N}\sum_{n=1}^N T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell \tag{2}$$

converge in the mean as $N \to \infty$.

Mean convergence is known when the transformations $T_1, \ldots, T_\ell$ are powers of the same transformation ([32, 35]), when the polynomials are linear [42] (with alternate proofs given in [43, 3, 29]), and when the polynomials have distinct degrees [15]. Convergence is also known for general families of polynomials if one imposes very strong ergodicity assumptions on the transformations [33]. See also [4, 5] where techniques from [3] have been refined and extended, aiming to eventually handle the case of general polynomial iterates. Despite such intense efforts, convergence is still not known for some simple families of polynomials, for instance, when $\ell = 2$ and $p_1(n) = p_2(n) = n^2$, or when $p_1(n) = n^2$, $p_2(n) = n^2 + n$.

As mentioned previously, when all transformations are equal, and the polynomials are essentially distinct, then characteristic factors of the averages (2) can be chosen to have very special algebraic structure. For general commuting transformations this is no longer the case; if one chooses $p_2 = p_1 = n$, $T_1 = T_2$, and $f_2 = \bar{f}_1$, then the averages (2) do not converge to 0 unless $f_1 = f_2 = 0$. The same problem persists when two of the polynomials are *pairwise dependent*, meaning, some non-trivial linear combination of two of the polynomials is constant. But in all other cases, there is no obvious obstruction to having "simple" characteristic factors.

Problem 6. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are* **pairwise** *independent. Show that there exists $d \in \mathbb{N}$ such that the factors $\mathcal{Z}_{d,T_1}, \ldots, \mathcal{Z}_{d,T_\ell}$ are characteristic factors for the averages* (2).

This is known to be the case when the polynomials have distinct degrees [15]. But it is not known for some simple families of integer polynomials, for instance, for the family $\{n^3, n^3+n\}$ or the family $\{n, n^2, n^2+n\}$. Even for weak mixing transformations the problem is open:

Special Case of Problem 6. *Suppose that the transformations $T_1, \ldots, T_\ell \colon X \to X$ are weak mixing and the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are* **pairwise** *independent. Show that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell = \int f_1 \, d\mu \cdot \ldots \cdot \int f_\ell \, d\mu.$$

When all transformations are equal and the polynomials are in general position, characteristic factors for the averages (2) turn out to be extremely simple [22, 24]:

Theorem. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent. Then the rational Kronecker factor $\mathcal{K}_{rat}(T)$*[1] *is a characteristic factor for the averages* (1).

It is very likely that this result generalizes to the case of several commuting transformations:

Problem 7. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent. Show that the factors $\mathcal{K}_{rat}(T_1), \ldots, \mathcal{K}_{rat}(T_\ell)$ are characteristic factors for the averages* (2).

This was proved in [15] when $\ell = 2$ and $p_1(n) = n$. In the same article a somewhat weaker property was proved for all monomials with distinct degrees. We mention also a closely related multiple recurrence problem:

Problem 8. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent and have zero constant term. Show that for every $A \in \mathcal{X}$ and every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T_1^{-p_1(n)} A \cap \cdots \cap T_\ell^{-p_\ell(n)} A) \geq \mu(A)^{\ell+1} - \varepsilon. \tag{3}$$

In fact, the set of integers n for which (3) holds is expected to have bounded gaps. The lower bounds are known when all transformations are equal [23] and they are also known for general commuting transformations when the polynomials are monomials with distinct degrees [15]. The result fails if the polynomials are distinct and pairwise dependent; in this case no fixed power of $\mu(A)$ works as a lower bound in (3) for every system and set [9]. On the other hand, the assumption that the polynomials are rationally independent is not necessary, for instance, the result is expected to work for the family of polynomials $\{n, n^2, n^2+n\}$ (this is known to be the case when all transformations are equal [21]). We remark that Problem 7 is solved, then the conjectured lower bounds of Problem 8 will follow rather easily.

Regarding pointwise convergence of multiple ergodic averages of commuting transformations, progress has been extremely scarce. Even when one uses two commuting transformations and linear iterates convergence is not known in general. The following is a well known open problem:

[1] Given a measure preserving system $(X, \mathcal{X}, \mu, T)$ we define $\mathcal{K}_{rat}(T) = \bigvee_{d\in\mathbb{N}} \mathcal{I}_{T^d}$.

Problem 9. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T, S\colon X \to X$ be commuting invertible measure preserving transformations, and $f, g \in L^\infty(\mu)$ be functions. Show that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n x) \cdot g(S^n x)$$

converge pointwise.

For a list of partial results that apply to special classes of transformations see the list after Problem 3.

1.3. **Not necessarily commuting transformations.** All problems in the previous sections were stated for families of transformations that commute. It is very likely that all positive results extend to the case where the transformations generate a nilpotent group. For instance, we mention a problem from [10]:

Problem 10. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T_1, \ldots, T_\ell\colon X \to X$ be invertible measure preserving transformations that generate a nilpotent group, and $f_1, \ldots, f_\ell \in L^\infty(\mu)$ be functions. Show that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell$$

converge in $L^2(\mu)$.

Convergence is known when $\ell = 2$ [10] and is open for $\ell = 3$. The interested reader should look in [10] for a list of other closely related open problems. See also [**?**] for a related multiple recurrence result.

When one works with arbitrary families of invertible measure preserving transformations the next result shows that one cannot expect to have similar convergence results:

Theorem. *Let $a, b\colon \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be $1-1$ sequences. Then there exist invertible Bernoulli measure preserving transformations T and S acting on the same probability space $(X, \mathcal{X}, \mu)$ such that*

- *for some $f, g \in L^\infty(\mu)$ the averages $\frac{1}{N}\sum_{n=1}^{N}\int T^{a(n)}f \cdot S^{b(n)}g \, d\mu$ diverge;*
- *for some $A \in \mathcal{X}$ with $\mu(A) > 0$ we have $T^{-a(n)}A \cap S^{-b(n)}A = \emptyset$ for every $n \in \mathbb{N}$.*

To construct such examples it suffices to modify examples of D. Berend (see Ex 7.1 in [6]) and H. Furstenberg (page 40 in [27]) that cover the case $a(n) = b(n) = n$ (the details will appear in [25]). When $a(n) = b(n)$, it is also known that given any finitely generated solvable group G of exponential growth, there exist invertible measure preserving transformations T, S, with $< T, S > \subset G$, and such that and the conclusion of the previous theorem holds for those T and S. It is interesting that despite such negative news, once one introduces an extra variable, several convergence (and very likely recurrence) results can be extended to arbitrary families of measure preserving transformations. We mention an example from [14]:

Theorem. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T_1, \ldots, T_\ell \colon X \to X$ be invertible measure preserving transformations, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ be functions, $p_1, \ldots, p_\ell$ be essentially distinct polynomials, and $a \in (0, 1/d)$. Then the averages*

$$\frac{1}{N^{1+a}} \sum_{1 \leq m \leq N, 1 \leq n \leq N^a} f_1(T_1^{m+p_1(n)} x) \cdot \ldots \cdot f_\ell(T_\ell^{m+p_\ell(n)} x) \tag{4}$$

converge pointwise as $N \to \infty$.

One can show that the assumption that the polynomials are essentially distinct is necessary. It was also shown in [14] that there exists $d \in \mathbb{N}$ such that the factors $\mathcal{Z}_{d,T_1}, \ldots, \mathcal{Z}_{d,T_\ell}$ are characteristic for pointwise convergence of the averages (4). Interestingly, the corresponding multiple recurrence result (that would generalize the polynomial Szemerédi theorem) remains open:

Problem 11. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T_1, \ldots, T_\ell \colon X \to X$ be invertible measure preserving transformations, and $p_1, \ldots, p_\ell$ be distinct polynomials with zero constant term. Show that for every $A \in \mathcal{X}$ with $\mu(A) > 0$ we have*

$$\mu(A \cap T_1^{-m-p_1(n)} A \cap \cdots \cap T_\ell^{-m-p_\ell(n)} A) > 0$$

for some $m, n \in \mathbb{N}$.[2]

The assumption that the polynomials are distinct is necessary since as mentioned before there exist (non-commuting) transformations T, S, acting on the same probability space $(X, \mathcal{X}, \mu)$, and a set $A \in \mathcal{X}$ with $\mu(A) > 0$ such that $\mu(T^n A \cap S^n A) = 0$ for every $n \in \mathbb{N}$. The multiple recurrence property is known to hold when all the transformations are weak mixing [14], but for general measure preserving systems even some of the simplest cases are open:

Special Case of Problem 11. *Let $(X, \mathcal{X}, \mu)$ be a probability space and $T, S, R \colon X \to X$ be invertible measure preserving transformations. Show that for every $A \in \mathcal{X}$ with $\mu(A) > 0$ there exist $m, n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-m} A \cap S^{-m-n} A \cap R^{-m-2n} A) > 0.$$

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[2] This would imply that given a countable amenable group G and arbitrary elements $a_1, \ldots, a_\ell \in G$, for every $E \subset G$ that has positive upper density with respect to some Følner sequence in G, there exist $g \in M$ and $m, n \in \mathbb{N}$ such that $g,\ a_1^{m+p_1(n)} g, \ldots, a_\ell^{m+p_\ell(n)} g \in E$.

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(Nikos Frantzikinakis) University of Crete, Department of mathematics, Knossos Avenue, Heraklion 71409, Greece

E-mail address: frantzikinakis@gmail.com