ADDITIVE FUNCTIONS MODULO A COUNTABLE SUBGROUP OF $\mathbb{R}$

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ABSTRACT. We solve the $\bmod G$ Cauchy functional equation

$$f(x+y) = f(x) + f(y) \pmod{G},$$

where G is a countable subgroup of $\mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is Borel measurable. We show that the only solutions are functions linear $\bmod G$.

1. Introduction.

It is well known that under any reasonable measurability assumption a function satisfies the Cauchy functional equation iff it is linear. We would like to extend this result for functions that satisfy the Cauchy functional equation modulo a countable subgroup G of the real numbers. More precisely, assuming that the Cauchy difference

$$f(x+y) - f(x) - f(y)$$

takes values in G, we would like to show that there exists $a \in \mathbb{R}$ such that the function

$$f(x) - ax$$

takes values in G. We will show that this is the case if f is Borel measurable.

Our motivation comes from a problem in ergodic theory. There, we want to solve in g the following functional equation

$$g(x+t) - g(x) = c(t) + h_t(x+a) - h_t(x) \pmod{1}, \tag{1}$$

where a is irrational, g and h_t are 1-periodic Lebesgue measurable functions, and (1) is valid for every t for some choice of h_t and $c(t)$. It turns out that the function $c(t)$ can be chosen to be Borel measurable and additive $\bmod G$, where G is the countable dense subgroup of the reals $\mathbb{Z} + \mathbb{Z}a$. Knowing that $c(t)$ is linear $\bmod G$ enables us to show that the solutions of (1) have the form

$$g(x) = mx + c + h(x+a) - h(x) \pmod{1}.$$

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The functional equation (1) was originally studied by Conze and Lesigne in a more general setup. It was actually solved in [2] but using a different argument than ours.

2. An extension result.

To facilitate the reading we quickly review some basic notions and facts from elementary topology. A subset of $\mathbb{R}^n$ (with the standard topology) is a *meager* (or *first category*) set if it is a countable union of nowhere dense sets. A set is of the *second category* if it is not meager. A *residual* set is the complement of a meager set. A property holds *quasi everywhere* (we write q.e.) in a set A, if it is true for all but a meager set of elements in A. A set has the *Baire property* if it is equal to the symmetric difference of an open set and a meager set. The space $\mathbb{R}^n$ is of the second category. All Borel subsets of $\mathbb{R}^n$ have the Baire property. Finally, the topological analogue of Fubini's theorem states that if E is a meager subset of $\mathbb{R}^{n+m}$ then for q.e. $x \in \mathbb{R}^n$ the x-section $E_x = \{y \in \mathbb{R}^m : (x, y) \in E\}$ is a meager subset of $\mathbb{R}^m$. Proofs for these facts can be found in [4].

A crucial step needed for the proof of our theorem is the extension result stated in the lemma below. Note that we do not need to assume any kind of measurability for this part. To prove it we use ideas from [1] and [3].

Lemma. *Suppose that the function $g \colon \mathbb{R} \to \mathbb{R}$ satisfies the Cauchy functional equation $g(x + y) = g(x) + g(y)$ q.e. in D, where $D = (a, b) \times (d, e)$ is a nonempty open rectangle. Then there exists a nonempty open interval $I \subset (a, b)$, a constant $c \in \mathbb{R}$, and an everywhere additive function $h \colon \mathbb{R} \to \mathbb{R}$, such that $g = h + c$, q.e. in I.*

Proof. We first assume that $D = I_r \times I_r$, where $I_r = (-r, r)$ for some $r > 0$. At the end we deal with the general case.

Our starting point is to define the function h. We claim that for every $x \in I_{\frac{r}{2}}$, the function $g(x + y) - g(y)$ is equal to a constant $h(x)$, y-q.e. in $I_{\frac{r}{2}}$. To see this, observe that our assumption and the topological version of Fubini's theorem show that there exists a set J_r, residual in I_r, such that for every $x \in J_r$

$$g(x + y) = g(x) + g(y), \quad y - \text{q.e. in } I_r. \tag{2}$$

Denote by $J_{\frac{r}{2}}$ the set $J_r \cap I_{\frac{r}{2}}$. The interval $I_{\frac{r}{2}}$ is centered at 0 and $J_{\frac{r}{2}}$ is residual in $I_{\frac{r}{2}}$, so for every $x \in I_{\frac{r}{2}}$ the sets $J_{\frac{r}{2}}$ and $x - J_{\frac{r}{2}}$ have nonempty intersection. Hence, for given $x \in I_{\frac{r}{2}}$ there exists $a(x) \in J_r$ such that

$x - a(x) \in J_{\frac{r}{2}}$. Applying (2) twice gives that for every $x \in I_{\frac{r}{2}}$ we have

$$\begin{aligned} g(x+y) &= g\big(a(x) + x - a(x) + y\big) \\ &= g\big(a(x)\big) + g\big(x - a(x) + y\big) \qquad && y - \text{q.e. in } I_{\frac{r}{2}} \\ &= g\big(a(x)\big) + g\big(x - a(x)\big) + g(y) \qquad && y - \text{q.e. in } I_{\frac{r}{2}}. \end{aligned}$$

Hence, for every $x \in I_{\frac{r}{2}}$ we have

$$g(x+y) - g(y) = g\big(a(x)\big) + g\big(x - a(x)\big) = h(x), \quad y - \text{q.e. in } I_{\frac{r}{2}}, \tag{3}$$

proving our claim.

Next we prove that h is additive everywhere in $D_{\frac{r}{4}} = I_{\frac{r}{4}} \times I_{\frac{r}{4}}$. So let $(u,v) \in D_{\frac{r}{4}}$. Our assumption combined with (3) give

$$g(u+w+v+z) = g(u+w) + g(v+z) \qquad (w,z) - \text{q.e. in } D_{\frac{3r}{4}}, \tag{4}$$

$$g(w+z) = g(w) + g(z) \qquad (w,z) - \text{q.e. in } D_r, \tag{5}$$

$$g(u+w) - g(w) = h(u) \qquad w - \text{q.e. in } I_{\frac{r}{2}}, \tag{6}$$

$$g(v+z) - g(z) = h(v) \qquad z - \text{q.e. in } I_{\frac{r}{2}}, \tag{7}$$

$$g(u+v+\xi) - g(\xi) = h(u+v) \qquad \xi - \text{q.e. in } I_{\frac{r}{2}}. \tag{8}$$

Since (4)-(7) hold q.e. there exist w and z in $I_{\frac{r}{4}}$ such that $\xi = w+z$ satisfies (8). For this choice of w and z we have

$$\begin{aligned} h(u+v) &= g(u+v+w+z) - g(w+z) && \text{(by (8))} \\ &= g(u+w) + g(v+z) - g(w+z) && \text{(by (4))} \\ h(u) + h(v) &= g(u+w) + g(v+z) - g(w) - g(z) && \text{(by (6),(7))} \\ &= g(u+w) + g(v+z) - g(w+z) && \text{(by (5))}. \end{aligned}$$

Thus,

$$h(u+v) = h(u) + h(v).$$

The choice of $(u,v) \in D_{\frac{r}{4}}$ was arbitrary, so h is additive everywhere in $D_{\frac{r}{4}}$, proving our claim.

Using a standard argument we extend h to an everywhere additive function in $\mathbb{R}$. (For example $h(x) = n\, h(\frac{x}{n})$, for $x \in n\, I_{\frac{r}{4}}$, defines unambiguously an everywhere additive function in $\mathbb{R}$.) Finally, for q.e. $x \in I_{\frac{r}{4}}$ we have

$$g(x+y) - g(y) = g(x), \quad y - \text{q.e. in } I_{\frac{r}{4}}.$$

So $h(x) = g(x)$ for q.e. $x \in I_{\frac{r}{4}}$. This completes the proof of the lemma in the special case where D is centered at the origin.

Now we deal with the general case, where $D = (a,b) \times (d,e)$. We would like to use the previous argument, so we define a function w that is additive q.e. in a neighborhood of the origin. Our assumption and Fubini's theorem give

$$g(u+v) = g(u) + g(v) \quad (u,v) - \text{q.e. in } D, \tag{9}$$

$$\text{for q.e. } u \text{ in } (a,b) \quad g(u+y) = g(u) + g(y) \quad y - \text{q.e. in } (d,e), \tag{10}$$

$$\text{for q.e. } v \text{ in } (d,e) \quad g(x+v) = g(x) + g(v) \quad x - \text{q.e. in } (a,b). \tag{11}$$

There exists $(u,v) \in D$ that satisfies all (9)-(11). For this choice of u and v choose r such that the rectangle $(u-r, u+r) \times (v-r, v+r)$ is contained in D, and consider the function w: $(-2r, 2r) \to \mathbb{R}$, defined by $w(s) = g(u+v+s) - g(u+v)$. We claim that for q.e. $s \in I_r$ we have $w(s+t) = w(s) + w(t)$, t-q.e. in I_r. To see this observe that (10) and (11) give that for q.e. $s \in I_r$

$$\begin{aligned} w(s+t) &= g(u+s+v+t) - g(u+v) \\ &= g(u+s) + g(v+t) - g(u+v), \quad t - \text{q.e. in } D_r. \end{aligned}$$

Moreover, by (9)-(11) we have that for q.e. $s \in I_r$

$$\begin{aligned} w(s) + w(t) &= g(u+s+v) + g(u+v+t) - 2g(u+v) \\ &= g(u+s) + g(v) + g(u) + g(v+t) - g(u) - g(v) - g(u+v) \\ &= g(u+s) + g(v+t) - g(u+v), \quad t - \text{q.e. in } D_r. \end{aligned}$$

Hence, for q.e. $s \in I_r$ we have

$$w(s+t) = w(s) + w(t), \quad t - \text{q.e. in } I_r,$$

proving our claim.

Next observe that we can apply the first part of the proof for the function w. Indeed, although our assumption here is weaker (we do not have additivity q.e. in D_r), it is all we need to carry out the previous argument. So there exists an everywhere additive function h such that $h(s) = w(s)$ for q.e. $s \in I_{\frac{r}{4}}$. We want to relate h with g. We write any $x \in (u - \frac{r}{4}, u + \frac{r}{4})$ as $x = u + s$, where $s \in I_{\frac{r}{4}}$. Using (9)-(11) we see that for q.e. s in $I_{\frac{r}{4}}$

$$\begin{aligned} h(u+s) &= h(u) + h(s) = h(u) + w(s) = h(u) + g(u+v+s) - g(u+v) \\ &= h(u) + g(u+s) + g(v) - g(u) - g(v) = g(u+s) + h(u) - g(u). \end{aligned}$$

Hence, $h(x) = g(x) + c$ for q.e. $x \in (u - \frac{r}{4}, u + \frac{r}{4})$, where $c = h(u) - g(u)$. This completes the proof of the lemma. $\square$

3. The mod G Cauchy functional equation.

We are now ready to prove our main result.

Theorem. *Let G be a countable subgroup of $\mathbb{R}$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is Borel measurable and satisfies the functional equation*

$$f(x+y) = f(x) + f(y) \pmod{G},$$

for all $x, y \in \mathbb{R}$. Then f is linear mod G.

Proof. The function $F(x,y) = f(x+y) - f(x) - f(y)$ is Borel measurable and countably valued (G is countable). Since $\mathbb{R}^2$ is of the second category F is equal to a constant c_1 in a second category Borel subset B of $\mathbb{R}^2$. The set B being Borel has the Baire property, so it is equal to a nonempty (B is of the second category) open set modulo a meager set. As a result, there exists a (nonempty) rectangle $D = (a,b) \times (c,d)$ that is contained in B up to a meager set. If we define $g(x) = f(x) + c_1$, we have

$$g(x+y) = g(x) + g(y), \quad \text{q.e. in } D.$$

By the Lemma there exists an additive function $h : \mathbb{R} \to \mathbb{R}$ and a nonempty open interval $I \subset (a,b)$, such that

$$g(x) = h(x) + c_2, \quad \text{q.e. in } I, \tag{12}$$

for some $c_2 \in \mathbb{R}$.

We claim that h is linear. To see this, observe that $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}(I + kl)$ where l is half the length of I. Since $h(x+kl) = h(x) + h(kl)$ we have

$$A_M = \{x \in \mathbb{R} : |h(x)| \le M\} = \bigcup_{k\in\mathbb{Z}} \{x \in I : |h(x) + h(kl)| \le M\}.$$

By (12) the last set is equal to the Borel set

$$\bigcup_{k\in\mathbb{Z}} \{x \in I : -M - h(kl) + c_2 \le g(x) \le M - h(kl) + c_2\}$$

up to a meager set. It follows that the set A_M has the Baire property. In addition, the union of the A_M's is $\mathbb{R}$, a second category set, so there exists $M_0 \in \mathbb{R}$ such that the set A_{M_0} is of the second category. Then A_{M_0} is equal to a nonempty open set U up to a meager set. Observing that $U + d$ intersects U non-trivially for small d, we conclude that the difference set $A_{M_0} - A_{M_0}$ contains a nonempty open interval centered at the origin. On this interval h will be bounded by $2M_0$, so we can use a standard argument to conclude that h is linear. This proves our claim.

So there exist $a, c_3 \in \mathbb{R}$ such that $f(x) = ax + c_3$ q.e. in I. The function f is additive mod G, so $c_3 \in G$. Moreover, an easy inductive argument gives

$$f(qx) = qf(x) \pmod{G'},$$

for every $q \in \mathbb{Q}$, where $G' = \{\frac{g}{n}, g \in G, n \in \mathbb{N}\}$. Since $\mathbb{R} = \bigcup_{q\in\mathbb{Q}} qI$ we get

$$f(x) = ax \pmod{G'}, \quad \text{q.e. in } \mathbb{R}.$$

Let C be the residual set where the last relation holds. Since C and $x - C$ have nonempty intersection for every $x \in \mathbb{R}$, every real number has the form $x = x_1 + x_2$, for some $x_1, x_2 \in C$. So

$$f(x) = f(x_1) + f(x_2) = ax_1 + ax_2 = ax \pmod{G'},$$

for every $x \in \mathbb{R}$.

It remains to show that we can replace G' by G in the last equation. For this we define $q(x) = f(x) - ax$ and claim that $q(\mathbb{R}) \subset G$. The function $q(x)$ is additive mod G, Borel measurable, and $q(\mathbb{R}) \subset G'$. For some $r \in G'$ the Borel set $q^{-1}(r)$ is of the second category. Choose $n \in \mathbb{N}$ such that $nr \in G$ and let $E = nq^{-1}(r)$. Then E is a second category Borel set and $q(E) \subset G$. Since q is additive we get $q(E - E) \subset G$. As before we see that $E - E$ contains a nonempty interval J centered at the origin. Then $q(J) \subset G$ and because G is a group we have that $n\,q(J) \subset G$ for every $n \in \mathbb{N}$. Since $q(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} q(nJ)$ and $q(nJ) = n\,q(J) \pmod{G}$ we get $q(\mathbb{R}) \subset G$, proving our claim. Thus,

$$f(x) = ax \pmod{G},$$

for every $x \in \mathbb{R}$. This completes our proof. $\square$

Remark. Suppose that the mod G Cauchy functional equation is satisfied for q.e. $(x, y) \in \mathbb{R}^2$. Then the previous argument applies and gives that every Borel measurable solution is equal q.e. to a linear function mod G.

Our argument does not seem to apply when f is assumed to be just Lebesgue measurable. The reason is that although every Borel measurable subset of $\mathbb{R}^2$ is equal to an open set modulo a meager set, not every Lebesgue measurable subset of $\mathbb{R}^2$ is equal to an open set modulo a set of measure zero. We leave the following question open.

Question. *Let G be a countable dense subgroup of $\mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function that is additive* mod G. *Is f necessarily linear* mod G*?*

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