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THE CONIC OF INTERSECTIONS OF AN AFFINITY

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Abstract In this article we study some conics defined, up to dilatation, by an affinity of the plane. We discuss the mutual relations between the conic and the defining affinity and, in particular, we show how to reproduce affinities related to a given conic. As an application of the theory, we show that the orbital conics of equiaffinities are dilatations of the considered conics.

1 Introduction

"Affinities" or *"Affine transformations"* and in particular affinities of the plane, we'll deal with here, are widely known and have been thoroughly studied ([7], [8, p.97], [4, p.191], [14]). It seems though, that the following conic λ (see Figure 1) directly and simply deriving from an affinity f, has not attracted much attention, neither its relations to the generating it affinity have been discussed. The conic is generated by considering a

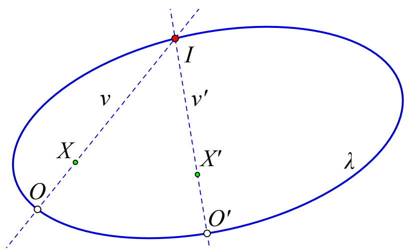

Figure 1: The intersection conic of an affinity

point O such that $O' = f(O) \neq O$ and the lines OX through O. The intersection points $\{I = OX \cap O'X'\}$ with the corresponding image-lines $O'X' = f(OX)$, as OX revolves about O, generate a conic. The conic apparently depends on the choice of the point O, but below (section 6) we show that the conics produced analogously for other choices of O are *dilatated* (in euclidean geometry terms: translated or homothetic) to λ or to the *conjugate* of λ in the case of hyperbolas. Thus, the affinity defines a certain conic *up to dilatation* and eventually conjugacy in the case of hyperbolas. We call this the affinity's *"conic of intersections"*.

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We should notice here the existence of a particular subgroup of affinities, which do not allow the definition of this conic. This is the subgroup of *"dilatations"* ([4, p.193]), which, by definition, map *every* line to a parallel line having an intersection point $I = OX \cap O'X'$ at infinity. In our context, these exceptional but simple affinities could be characterized by having their conic λ coinciding with the *line at infinity*. In the sequel we'll deal with affinities other than dilatations, that define genuine or, in some cases, degenerated (products of two lines) conics λ.

Regarding the organization of the material, in sections 2 and 3 we review in short the basic facts about affinities, their fixed points and their invariant lines. In sections 4, 5 and 6 we introduce and discuss the basic properties correspondingly of the *"associated ellipse"* and the *"conic of intersections"* of an affinity. In section 7 we discuss the existence of affinities having a given conic as their conic of intersections. In section 8 we examine the image $\lambda' = f(\lambda)$ via the affinity f of its conic of intersections λ. In sections 9 and 10 we construct affinities without fixed points having a given intersection conic. Finally in section 11 we apply the results of the preceding sections to *"equi-affinities"*, i.e. affinities preserving the area of triangles, to show that an *"orbital conic"* of such an affinity, i.e. the conic containing a sequence $\{X, f(X), f^2(X), ...\}$ for some point X, is a dilatation of its conic of intersections.

2 Affinities of the plane

Geometrically the affinities can be described as those invertible transformations of the plane onto itself, which transform lines to lines ([15, vol. II], [16, III, p.18], [1]). Their group contains as a subgroup the *"euclidean isometries"*, which preserve the distances of points $|X'Y'| = |XY|$ and the *"similarities"*, which multiply distances by a constant $|X'Y'| = k|XY|$ ([16, vols I, II], [1]). Analytically, fixing a coordinate system of the plane (not necessarily orthogonal or having equal unit-lengths on the axes), the affinities are described by an invertible matrix $\{A,\ |A| = a_{11}a_{22} - a_{12}a_{21} \neq 0\}$ and a vector $v(v_1, v_2)$ ([4, p.203]):

$$Y = f(X) = AX + v, \quad \text{with} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} :$$

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + v_1, \\ y_2 &= a_{21}x_1 + a_{22}x_2 + v_2. \end{aligned}\right\} \tag{1}$$

Using the three dimensional extensions $X' = (x_1, x_2, 1)$ of points $X(x_1, x_2) \in \mathbb{R}^2$ and the matrix

$$A_v = \begin{pmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ 0 & 0 & 1 \end{pmatrix} = \left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right), \tag{2}$$

the representation (1) of the affinity is equivalent with

$$Y' = A_v \cdot X' \quad \Leftrightarrow \quad \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = A_v \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}. \tag{3}$$

The determinants of A_v and A, are *equal* and the inverse of A_v is of the same form:

$$A_v^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} a_{22} & -a_{12} & a_{12}v_2 - a_{22}v_1 \\ -a_{21} & a_{11} & a_{21}v_1 - a_{11}v_2 \\ 0 & 0 & |A| \end{pmatrix} = \left(\begin{array}{c|c} A^{-1} & -A^{-1} \cdot v \\ \hline 0 & 1 \end{array}\right).$$

Also, the product (composition) of two such transformations, represented by the matrices $\{A_v, B_w\}$, is of the same form:

$$\left(\begin{array}{c|c} A & v \\ \hline 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} B & w \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} AB & Aw+v \\ \hline 0 & 1 \end{array}\right).$$

Remark 1. From the coordinates (x, y) we pass to $(x, y, 1)$ and from there to the corresponding *"projective coordinates"* (x, y, z), in which $\{(x, y, z), (kx, ky, kz),\ k \neq 0\}$ represent the same point $(x/z, y/z, 1)$.

The lines $ax + by + c = 0$ are described in projective coordinates by $ax + by + cz = 0$ and $z = 0$ represents the *"line at infinity"* containing the far out points $\{(x, y, 0)\}$ of the plane. From equation (1) we see that affinities map a point $(x, y, 0)$ to a point of the same kind $(x', y', 0)$ i.e. *affinities preserve the line at infinity*, consequently, ordinary points of the plane, characterized by coordinates of the form $(x, y, 1)$, map by affinities to points of the same kind $(x', y', 1)$.

This has an important consequence concerning the image of a circle by an affinity. This is a conic, since quadratic equations transform to quadratic equations by affinities. This conic though cannot have points at infinity and *the images of circles via affinities are ellipses*.

The affine geometry ([4, p.191], [6, p.98]) of the plane deals with properties of shapes that remain the same (invariant) when the shapes are transformed by affinities. For convenience of reference, next theorem lists the most elementary of these properties, making subsequently some comments but without entering into their proofs, which can be found in any one of the aforementioned references.

Theorem 1. *The following are valid properties for any affinity f of the plane. Two shapes $\{S, S' = f(S)\}$ related by an affinity are said "affine equivalent".*

1. [collinearity property] f *maps collinear points to collinear and non-collinear to non-collinear points.*
2. [equivalence of triangles] *Any two triangles $\{ABC, A'B'C'\}$ are affine equivalent by a unique affinity mapping $\{A \mapsto A', B \mapsto B', C \mapsto C'\}$. In particular, an affinity fixing the vertices of a triangle is the identity transformation.*
3. [parallels preservation] *An affinity maps parallel lines to parallels.*
4. [ratio preservation] *The signed ratio $k = AB/BC$ of three points on a line is the same with that of their image points under an affinity $A'B'/B'C' = k$. In particular, affinities preserve the middles of segments.*
5. [areas quotient preservation] *The quotient of areas $a(S')/a(S)$ of a shape S and its image $S' = f(S)$ for a given affinity f is the same for all shapes S.*
6. [centroid preservation] *Affinities preserve the centroid $X_0 = (1/n)(X_1 + \cdots + X_n)$ of any finite set of points $\{X_1, \ldots, X_n\}$ of the plane.*
7. [equivalence of parallelograms] *Any two parallelograms $\{ABCD, A'B'C'D'\}$ are affine equivalent by an affinity mapping $\{A \mapsto A', B \mapsto B', C \mapsto C', D \mapsto D'\}$.*

Regarding the analytical point of view of these properties we notice, that given three points of the plane $\{X, Y, Z\}$, the determinant of the corresponding matrix

$$XYZ = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{pmatrix}, \quad |XYZ| := (y_1z_2 - y_2z_1) + (z_1x_2 - z_2x_1) + (x_1y_2 - x_2y_1),$$

if the coordinate frame is *"orthonormal"*, expresses twice the signed area of the triangle XYZ ([13, p.239]). Otherwise, in an *oblique* coordinate frame, it is a constant multiple of this area, the constant factor depending on the frame. This implies that the three points are *collinear* precisely when this determinant vanishes. Also applying the affinity to $\{X, Y, Z\}$ we obtain three other points $\{U, V, W\}$ and we can describe this operation with a matrix multiplication:

$$UVV = A_v \cdot XYZ \quad \Rightarrow \quad |UVW| = |A_v| \cdot |XYZ| = |A| \cdot |XYZ|.$$

This implies that if $\{X, Y, Z\}$ are non-collinear(collinear) the same is true for $\{U, V, W\}$. In addition, it follows that the quotient of the areas of two triangles is preserved by affinities. By splitting a polygon in triangles, we conclude that affinities preserve the quotient of areas of two polygons. There is even a kind of affinities *preserving* the area. This, by the last formula, happens when the determinant of the matrix is $|A| = |A_v| = 1$. These special affinities are called *"equiaffinities"* and have attracted much attention in the past ([15, II, p.105], [4, p.203], [3]). In sections 8 and 11 we discuss some properties of these transformations relating them to the conics under consideration.

The inverse of the matrix XYZ, for non-collinear points, is found to be

$$(XYZ)^{-1} = \frac{1}{|XYZ|} \begin{pmatrix} y_2 - z_2 & z_1 - y_1 & y_1 z_2 - y_2 z_1 \\ z_2 - x_2 & x_1 - z_1 & x_2 z_1 - x_1 z_2 \\ x_2 - y_2 & y_1 - x_1 & x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

Thus, given two triples of *non-collinear* points $\{(X, Y, Z),\ (U, V, W)\}$ the matrix equation

$$B \cdot XYZ = UVW \quad \Leftrightarrow \quad B = UVW \cdot (XYZ)^{-1},$$

has a unique solution and defines the matrix B, which, a short calculation shows to be of the form of equation (2), thus defining an affinity.

Remark 2. The matrix A controls the behavior of the points at infinity $X' = (x, y, 0)$ since the affinity acts on these points by ignoring the vector part: $A_v X' = AX$, where $X = (x, y)$. Characteristically, in the case of a dilatation, A must operate on every (x, y) as multiplication with a scalar (as an eigenvector), hence it is a scalar multiple of the identity matrix, and every dilatation can be represented in the form $Y = k \cdot X + v$ for a constant scalar k.

3 Fixed points, invariant lines

An affinity f may possess a center, i.e. an *"isolated"* point $X(x, y)$ such that

$$X = f(X) = AX + v, \quad \text{with} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\left.\begin{aligned} x_1 &= a_{11}x_1 + a_{12}x_2 + v_1, \\ x_2 &= a_{21}x_1 + a_{22}x_2 + v_2. \end{aligned}\right\} \tag{4}$$

This is a linear system of equations and it has a unique solution when its determinant is non zero:

$$D = \begin{vmatrix} a_{11} - 1 & a_{12} \\ a_{21} & a_{22} - 1 \end{vmatrix} = |A| - tr(A) + 1 \neq 0, \tag{5}$$

where $|A|$ is the determinant of A and $tr(A) = a_{11} + a_{22}$ is its *"trace"*. If condition (5) is satisfied, then the solution is

$$x_0 = \frac{1}{D}(a_{12}v_2 + (1 - a_{22})v_1) \ , \ y_0 = \frac{1}{D}(a_{21}v_1 + (1 - a_{11})v_2) \, . \tag{6}$$

The condition $D = |A| - tr(A) + 1 = 0$ characterizes the lack of fixed points, as in the case of *"translations"*, or the existence of a whole line of fixed points of the affinity, as in the case of *"shears"*. In fact if this condition holds, which means that 1 is an eigenvalue of A, then the system (4) either has no solution or has a whole line of points $X = X_0 + tw$ satisfying it. In the latter case w must be an eigenvector to the eigenvalue 1 of A. This is immediately seen by applying the transformation:

$$\begin{aligned} AX + v = X \quad &\Leftrightarrow \quad A(X_0 + tw) + v = X_0 + tw \\ &\Leftrightarrow \quad (AX_0 + v - X_0) + t(Aw - w) = 0 \quad \Rightarrow \quad Aw = w, \end{aligned}$$

since the before to last equation is true for $t = 0$ and $t = 1$.

Corollary 1. *If the affinity f possesses an "axis" of fixed points, then the matrix A has 1 as eigenvalue and also holds:*

$$|A| - tr(A) + 1 = 0 \, . \tag{7}$$

Corollary 2. *The affinity f has a unique fixed point, if and only if, 1 is not an eigenvalue of its matrix.*

Remark 3. The *"generic"* affinity satisfies $|A| - tr(A) + 1 \neq 0$ and consequently has a center. If the affinity f has no fixed point, then taking any point X of the plane with $X' = f(X)$ and setting $v = X - X'$ the composition $f' = f \circ T_v$, where T_v is the translation by v, has X' as center. Thus, any affinity either has a fixed point or its composition f' by an appropriate translation has a fixed point.

An affinity may also possess *"invariant lines"* i.e. lines which map to themselves. Assuming the invariant line ε parameterized in the form $\{\varepsilon(t) = X_0 + tw\}$ the condition $f(\varepsilon) = \varepsilon$ implies $A(X_0 + tw) + v = X_0 + t'w$ for a t' depending linearly on t : $t' = \alpha t + \beta$. This leads to the relation:

$$AX_0 + v - (X_0 + \beta w) = t(\alpha w - Aw) \, .$$

On the left side all is constant and on the right side we have something variable, depending on t. This implies the two conditions:

$$Aw = \alpha w \quad \text{and} \quad AX_0 + v = f(X_0) = (X_0 + \beta w) \, ,$$

showing that the direction w of the invariant line is an eigenvector of the matrix A corresponding to the eigenvalue α.

Corollary 3. *Any point-wise non-fixed but invariant line ε of an affinity f either has a unique fixed point or f acts as a translation on it.*

Proof. Using the preceding remarks, a point $X_0 + tw \in \varepsilon$ fixed by f will satisfy

$$\begin{aligned} A(X_0 + tw) + v = X_0 + t'w = X_0 + (\alpha \cdot t + \beta)w = X_0 + tw \quad \Rightarrow \\ t = \alpha \cdot t + \beta \quad \Leftrightarrow \quad t = \frac{\beta}{1 - \alpha}, \ \text{if } \alpha \neq 1. \end{aligned}$$

If $\alpha = 1$ we see easily that $f(X_0 + tw) - (X_0 + tw) = \beta \cdot w$, showing that f acts on ε as a translation by the vector $\beta \cdot w$. □

4 The associated ellipse

As we noticed in section 2, the affine image of any circle is an ellipse carrying some information about the affinity, and in some cases reflecting completely its behavior. According to the following lemma all these ellipses obtained from the various circles of the plane are related.

Lemma 1. *The images of the various circles under the same affinity are pairwise dilatated ellipses.*

Proof. The reason for this is illustrated in figure 2 for the case of a homothety. The conjugate $h' = f \circ h \circ f^{-1}$ of a homothety h or a translation by an affinity f is respectively a homothety or a translation. In fact, a homothety is represented in cartesian coordinates by the equation $Y = kX + (1-k)X_0$, where X_0 is the center of the homothety and k its ratio. It is then readily seen that $f \circ h \circ f^{-1}$ is represented by $Y = kX + (1-k)(AX_0 + v)$, which is also a homothety with the same ratio and center at the point $Y_0 = AX_0 + v = f(X_0)$. Analogously is seen that the conjugate $f \circ t_v \circ f^{-1}$ of a translation by the vector v is the translation by the vector Av. Since any two circles are the translate or homothetic by the ratio $k = r/r'$ of their radii, the claim is true and the resulting ellipses are homothetic by the same ratio k, or in the case of translation, each is a translation of the other. □

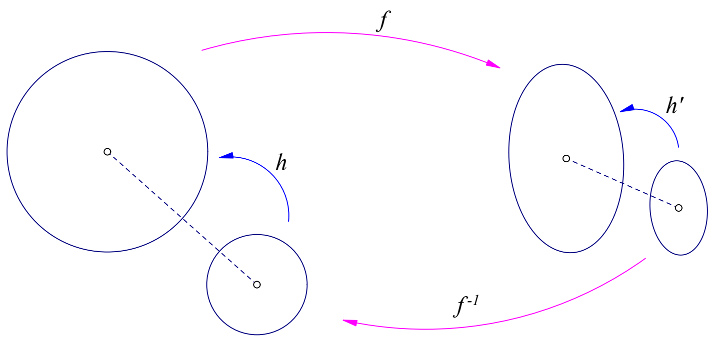

Figure 2: Affine images of circles

Thus, an affinity defines an infinity of pairwise dilatated ellipses and by abstracting to the proper shape and orientation of such an ellipse we can speak of the *"ellipse associated to an affinity"*. We can represent this ellipse by identifying it with the image via f of the unit circle $x^2 + y^2 = 1$, the coordinates now assumed to be ordinary cartesian referred to orthogonal vectors of length 1. Assuming the affinity f in the form

$$\left.\begin{aligned} x' &= ax + by + v_1, \\ y' &= cx + dy + v_2, \end{aligned}\right\} \Rightarrow \left.\begin{aligned} Dx &= dx' + -by' + bv_2 - dv_1, \\ Dy &= -cx' + ay' + cv_1 - av_2, \end{aligned}\right\}, \; D = ad - bc. \tag{8}$$

A matrix representation of the ellipse is obtained through the product of matrices:

$$N = M^t \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot M \text{ with } M = \begin{pmatrix} d & -b & bv_2 - dv_1 \\ -c & a & cv_1 - av_2 \\ 0 & 0 & ad - bc \end{pmatrix}, \tag{9}$$

M being the $D-$multiple of the inverse of the matrix representing f and M^t denoting the transposed of M. Doing the multiplication we find:

$$N = \begin{pmatrix} A & B & Av_1 + Bv_2 \\ B & C & -Bv_1 - Cv_2 \\ Av_1 + Bv_2 & -Bv_1 - Cv_2 & Av_1^2 + 2Bv_1v_2 + Cv_2^2 + D^2 \end{pmatrix}, \tag{10}$$

with $\{A = -c^2 - d^2 \ , \ B = ac + bd \ , \ C = -a^2 - b^2\}$. The invariants ([11]) of this conic are

$$J_1 = A + C \quad , \quad J_2 = AC - B^2 = D^2 \quad , \quad J_3 = det(N) = D^4 \ .$$

The center of the ellipse is the image $(v_1, v_2) = f(0,0)$ and translating there the coordinate system, the equation of the ellipse obtains the next form, in which the coefficients do not contain the constants $\{v_1, v_2\}$:

$$Ax^2 + 2Bxy + Cy^2 + D^2 = 0 \ .$$

A trivial example of characterization of the affinity by its associated ellipse is the case of the *"similarity"*, for which the associated ellipse is actually a circle. It is also not difficult to see, that if the associated ellipse is a circle, then the affinity is a similarity.

Remark 4. On the occasion of this family of pairwise dilatated ellipses we say a few words on the more general phenomenon of families of pairwise dilatated conics and the analytic representation of such families. It is well known ([6, p.112], [12, p. 273]), that fixing a cartesian system of coordinates and considering the corresponding quadratic equation $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ representing a conic, the quadratic coefficients of two dilatated conics are proportional $\left(\frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C}\right)$. The inverse is not quite true and we need some additional information on the two conics, if we have such a proportionality condition. For example, equations $x^2 - y^2 = 1$ and $x^2 - y^2 = 0$ or $x^2 - y^2 = -1$ have pairwise proportional quadratic coefficients but each is not a dilatation of the other in the strict sense of the concept. In order to conclude about the dilatation, we need some additional information: (i) whether the two conics are genuine or degenerate, (ii) in the case of hyperbolas, brought to their normal form, whether the signs of the constant terms are the same. If they are not the same, then the equations represent two hyperbolas, each similar to the conjugate of the other. In the sequel we'll use the term *"dilatated"* in this broader sense, including the similarity to the conjugate of a hyperbola. In figure 3 the two

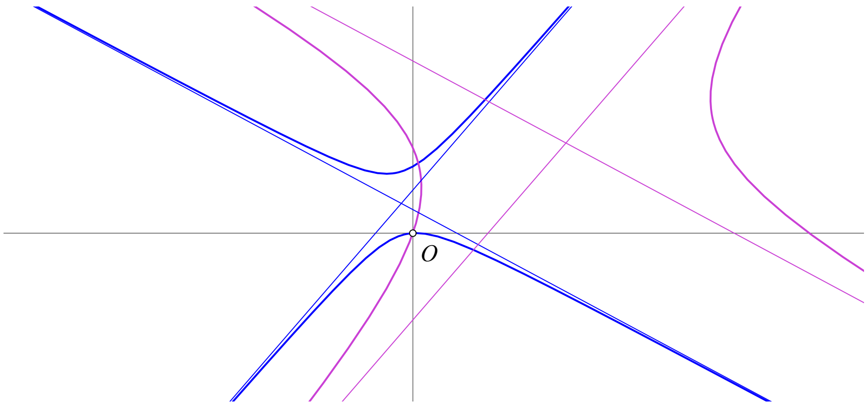

Figure 3: Dilatated conics in a broader sense

hyperbolas pass through the origin of coordinates and are defined by equations having the *same* quadratic coefficients and differing only in the linear terms. Their characteristic is that they have parallel respective asymptotes. Varying the coefficients of the linear terms we can produce such examples, where each hyperbola is similar to the conjugate hyperbola of the other.

5 The conic of intersections of an affinity

Figure 4 shows a circle $\kappa(O)$ and its image, the ellipse κ', under the affinity f. It also shows the conic λ produced by the intersections $\{I = OX \cap O'X'\}$ as OX revolves about the point O and its image $O'X' = f(OX)$ revolves about $O' = f(O) \neq O$.

Theorem 2. *For an affinity f different from a dilatation and a point O with $O' = f(O) \neq O$, the intersections $I = OX \cap O'X'$ of lines OX rotating about the point O and their images $O'X'$ generate a conic λ passing through the points $\{O, O'\}$. The conic passes also through the isolated fixed point D of the affinity, if any. The tangent μ at O maps via f to line $\mu' = OO'$ and this, in turn, maps to the tangent μ'' at O'.*

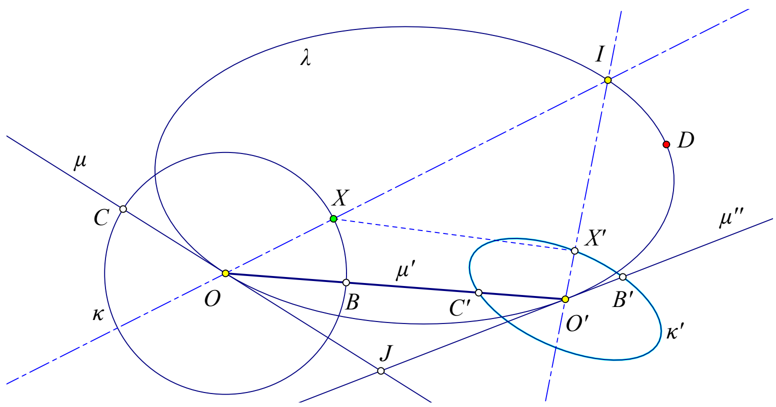

Figure 4: The ellipse $\kappa' = f(\kappa)$ and the conic λ of the affinity f

Proof. In fact, from the *"Chasles-Steiner theorem"* ([5, p.77]), lines such as $\{OX, OX'\}$ of two pencils $\{O^*, O'^*\}$ corresponding under a homography have an intersection I describing a conic passing through the centers $\{O, O'\}$ of the pencils. The statement about D is obvious and the statements about the tangents follow by considering the limiting positions of the lines $\{OX, O'X'\}$, when I tends to coincide with O or O'. □

We call λ the *"conic of intersections"* of the affinity f. Below we show that the shape of this conic is in some sense independent of the choice of the point O with $O' = f(O) \neq O$. More precisely, the various choices of such points O lead to dilatated to λ conics and when λ is a hyperbola to dilatated to λ or dilatated to its conjugate hyperbola λ^*.

Before to handle this question, we take a look at the *degenerate cases* i.e. the cases in which the conic λ is degenerate, consisting of two different lines, parallel or intersecting, the case of *"double line"* being excluded, since in that case the affinity would map the whole plane onto that line, which is not allowed. On the ground of figure 4, a key observation is, that if I coincides with a fixed point D, then, by the preservation of ratios by affinities, $OX/XD = O'X'/X'D$ and line XX' is parallel to OO'. If the conic λ contains a second fixed point $D' \neq D$ of f, then the whole line DD' consists of fixed points and is contained in λ, which therefore is degenerate, consisting of the union of the two lines $\{OO', DD'\}$ and for all X the line XX' is parallel to OO'. The possible configurations correspond to (i) $OO' \cap DD' = \emptyset$ and (ii) $OO' \cap DD' \neq \emptyset$ (see Figure 5). In both cases joining the arbitrary point X with O and defining $D_X = OX \cap DD'$, by the preservation of the ratios, we obtain $X' = f(X)$ on the line $D_X O'$ with $XX' \parallel OO'$. In coordinates with x-axis the line DD', the transformations are respectively described, each by a corresponding constant

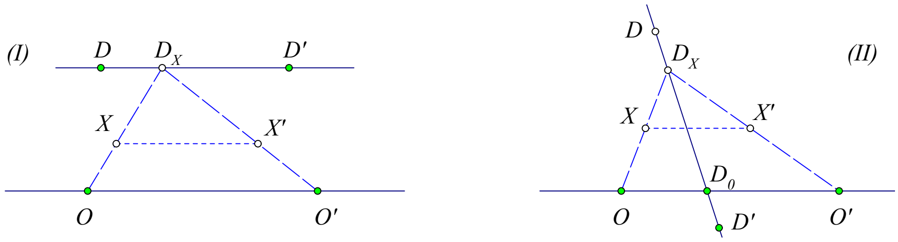

Figure 5: Affinities with degenerate conic of intersections

k, through the equations:

$$\text{"shear"}: \left\{ \begin{array}{lcl} x' & = & x + ky, \\ y' & = & y. \end{array} \right\} \quad \text{and} \quad \text{"strain"}: \left\{ \begin{array}{lcl} x' & = & x\,, \\ y' & = & ky\,. \end{array} \right\}$$

These two kinds of affinities, the *"shears"* and *"strains"*, are called *"homologies"* or *"axial affinities"* ([4, p.203], [14, p.116]). They are characterized as the affinities possessing a single line of fixed points, their *"axis"*, the constant k being the *"ratio"* of the homology. In the case of strains the direction of the lines $\{OO'\}$ is called *"conjugate direction"* of the strain. When $k = -1$, the affinity is called *"affine reflection"* ([4, p.203]). When the conjugate direction of the affine reflection is orthogonal to its axis we have a usual euclidean reflection. Obviously, keeping the same affinity f and changing the location of point O produces an other degenerate conic consisting of the same axis (DD') of fixed points and a different line O_1O_1', parallel to OO', so that in any case the resulting conics of intersections are, each a dilatation of the other. It is also easy to show the converse, i.e. that any axial affinity, shear or strain, and any choice of a point O with $O' = f(O) \neq O$ produces a degenerate conic of one of these two kinds. We formulate these observations in the form of a theorem.

Theorem 3. *Axial affinities produce degenerate conics of intersections and vice versa. These conics consist of two parallel lines in the case of shears and of two intersecting lines in the case of strains.*

Figure 6 shows the conic of intersections when f is a rotation $D(\varphi)$. It is easily seen, that in this case λ coincides with the circle through $\{O, O' = f(O), D\}$. Below we'll see

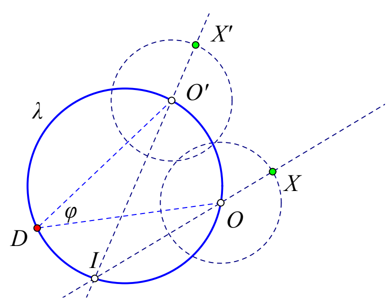

Figure 6: Rotation: the conic of the affinity is a circle

more general (corollary 4), that conversely, if the conic of intersections λ of the affinity is a circle, then the affinity is a *"spiral"* or *"direct similarity"* ([16, v.II]).

Remark 5. Notice that points $\{O, O'\}$ can be used to define also the conic of intersections of the inverse affinity $g = f^{-1}$. In this case their roles are interchanged and $O = g(O')$ but the conics of intersections of f and g coincide.

6 Independence from the base point

The conic λ of theorem 2 is in some sense independent of the selected point O. Next theorem discusses this issue assuming that the intersection conics are non-degenerate, since the analogous result for degenerate conics has already been handled in section 5. Also the term *"dilatation"* is used in the broader sense of remark 4.

Theorem 4. *With the conventions adopted so far, let $\{O, O_1\}$ be two non-fixed points of the non-axial affinity f. Let also $\{\lambda, \lambda_1\}$ be the corresponding non-degenerate conics of intersections. Then, λ_1 is a dilatation of λ.*

Proof. Let λ be generated by the intersections $\{I = \nu \cap \nu'\}$ of lines ν revolving about O and their images $\nu' = f(\nu)$ revolving about $O' = f(O)$. For each position of ν we consider its parallel ν_1 from O_1 and its image $\nu_1' = f(\nu)$ revolving about $O_1' = f(O_1)$, the intersections $\{I_1 = \nu_1 \cap \nu_1'\}$ generating the conic λ_1 (see Figure 7).

Consider a cartesian system of coordinates with origin at O and a variable unit vector $e(c, s)$. The lines ν through O are described by $\{te\ ,\ t \in \mathbb{R}\}$. If the affinity is represented by the matrix A and the translation vector v by $x' = Ax + v$, then the image lines ν' are described by $\{A(te) + v\ ,\ t \in \mathbb{R}\}$ and their intersections result from the equation

$$te \ = \ A(t'e) + v \quad \Rightarrow \quad I(e(c,s)) \ = \ \frac{v \cdot J(Ae)}{e \cdot J(Ae)} e \quad \Rightarrow \quad (x, y) \ = \ \frac{Mc + Ns}{Pc^2 + Qcs + Rs^2}(c, s)\ ,$$

where $J(x)$ is the positive rotation by $\pi/2$ and $x \cdot y$ represents the standard inner product, the last expression being the expansion in the coordinates (c, s). The coefficients $\{P, Q, R, M, N\}$ result by expanding the expressions in the fraction w.r.t. $\{c, s\}$. Eliminating $\{c, s\}$ we find the equation of λ :

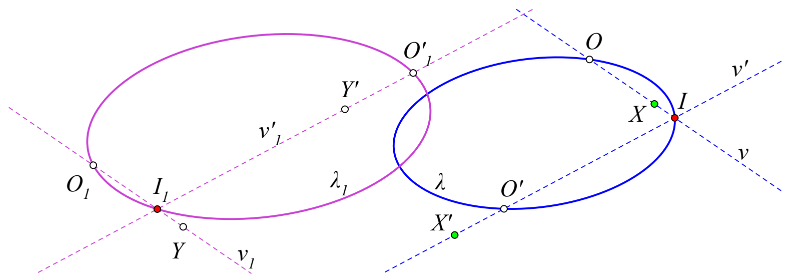

Figure 7: Dilatated conics generated by affinities

$$\lambda(x, y) \ = \ Px^2 + Qxy + Ry^2 - Mx - Ny \ = \ 0\,.$$

Analogously, the parallel to ν lines ν_1 through O_1 are $\{O_1 + te\ ,\ t \in \mathbb{R}\}$ and their images ν_1' are $\{A(O_1 + te) + v\ ,\ t \in \mathbb{R}\}$. From this results the representation of $I_1(e)$:

$$I_1(e(c,s)) - O_1 \ = \ \frac{(AO_1 + v - O_1) \cdot J(Ae)}{e \cdot J(Ae)} e \ = \ \frac{M_1 c + N_1 s}{Pc^2 + Qcs + Rs^2}(c, s)\,.$$

Disregarding the translation by O_1 and maintening in the following the symbol λ_1 for the translated conic, this leads by elimination of $\{c, s\}$ to the equation:

$$\lambda_1(x, y) \ = \ Px^2 + Qxy + Ry^2 - M_1 x - N_1 y \ = \ 0\,.$$

The conclusion follows from the well known facts discussed in remark 4. $\square$

7 Affinities with given conic of intersections

A given conic λ can be considered the conic of intersections of a multitude of affinities. Here we examine a method to define such affinities, which uses the constructs of the preceding section. In this section we assume that the affinity has an isolated fixed point. The case of affinities without fixed points will be examined in subsequent sections (9, 10). We start with a given ordered triple of pairwise different points $\{O, O', D\}$ of the conic, assumed to satisfy $\{f(O) = O', f(D) = D\}$. The standard method of definition of an affinity f proceeds by detecting two triangles $\{ABC, A'B'C'\}$ whose vertices are supposed to correspond under f : $\{A \overset{f}{\mapsto} A', B \overset{f}{\mapsto} B', C \overset{f}{\mapsto} C'\}$.

Our method uses an alternative procedure, replacing some of the points with lines. We'll proceed to the construction of f after a short discussion on this alternative method. In fact, we can more generally ask, *"if giving three elements and their images define an affinity?"*. Here *"elements"* stands for *points* or *lines*. Obviously giving three lines $\{\lambda, \mu, \nu\}$ in general position and their images $\{\lambda', \mu', \nu'\}$ also in general position, defines completely an affinity. In fact, considering the intersections of pairs of these lines, this reduces to the *standard* case of three points and their images. Next case leads also to a complete determination.

Theorem 5. *Two triples $\{(A, \mu, \nu), (A', \mu', \nu')\}$ of a point and two lines in general position uniquelly determine an affinity f mapping the first to the second.*

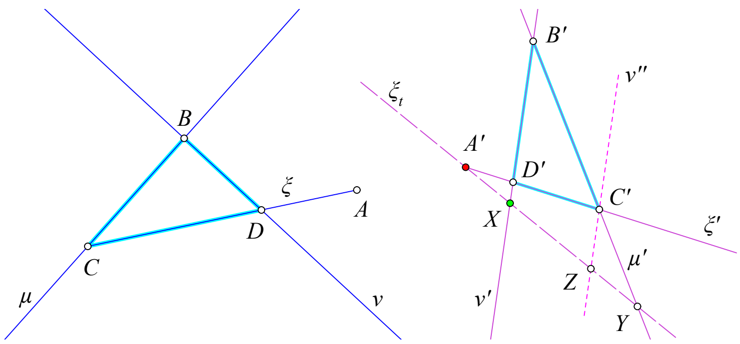

Figure 8: Affinity defined by a point and two lines and their images

Proof. Suppose we have succeeded in constructing the affinity f. Consider the intersections of an arbitrary line through A : $\{\xi\ :\ C = \xi \cap \mu\ ,\ D = \xi \cap \nu\}$ (see Figure 8) and the points $B = \mu \cap \nu,\ \ B' = \mu' \cap \nu'$. Certainly $f(B) = B'$ and to reduce the construction to that of three points and their images, it suffices to find the intersections $C' = \xi' \cap \mu'$, $D' = \xi' \cap \nu'$, where $\xi' = f(\xi)$. Since the ratio $k = AD/DC$ is known, it suffices to find the line ξ' through A' which intersects $\{\mu', \nu'\}$ at points $\{C', D'\}$ such that $A'D'/D'C' = k$. But this is easily achieved by taking an arbitrary point $X \in \nu'$ and drawing the parallel ν'' to ν' such that the variable lines ξ_t through A' intersect it at points Z such that $A'X/XZ = k$. The requested point $C' = \nu'' \cap \mu'$ and $D' = \nu' \cap A'C'$. □

Next theorem shows that the situation is radically different if we are given two points and a line and their images.

Theorem 6. *Two triples $\{(A, B, \mu), (A', B', \mu')\}$ of two points and a line in general position either fail to determine an affinity mapping the first to the second or they define a double infinity of affinities f doing that transformation.*

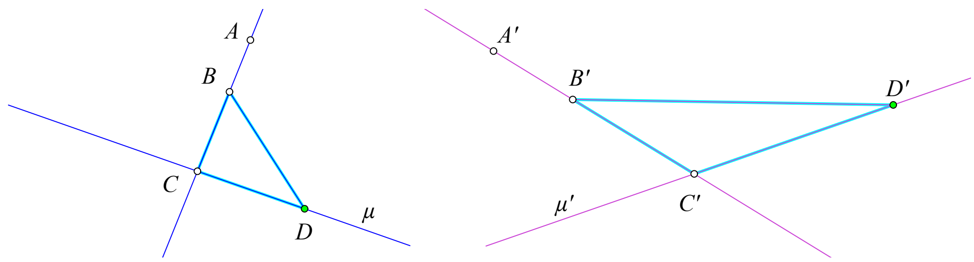

Figure 9: Affinities defined by two points $\{A, B\}$ and a line μ and their images

Proof. Suppose we have again succeeded in constructing the affinity f (see Figure 9). By the preservation of ratios by affinities, if $\{C = AB \cap \mu\ ,\ C' = A'B' \cap \mu'\}$ the ratios must be equal $AB/BC = A'B'/B'C' = k$. Thus, if this condition is not satisfied by the given data, there is no affinity mapping the first triple onto the second.

On the other side, if this condition is satisfied, then selecting two arbitrary points $\{D \in \mu\ ,\ D' \in \mu'\}$, we see easily that the affinity mapping BCD onto $B'C'D'$ satisfies the requirements. □

We proceed now to the definition of an affinity from a non-degenerate conic λ and three points on it, using the construction of theorem 5.

Theorem 7. *Consider an ordered triple of points $\{O, O', D\}$ on a non degenerate conic λ. There is a unique affinity f mapping $f(O) = O'$, leaving the point D fixed and having the conic λ as a conic of intersections.*

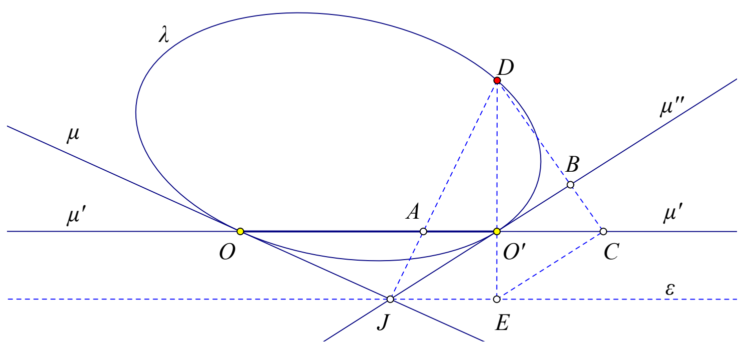

Figure 10: Affinity defined by three points on a conic

Proof. We use the additional information (theorem 2) of section 5 resulting directly from the proper definition of λ. The two points $\{O, O'\}$ define a chord μ' and two tangents $\{\mu, \mu''\}$ at its endpoints (see Figure 10). By theorem 5 there is an affinity f mapping the triple $\{\mu, \mu', D\}$ to $\{\mu', \mu'', D\}$. It is then easily seen that the given conic λ is a conic of intersections of this affinity. In fact, if λ' is the conic of intersections of f, then, by theorem 2, the lines $\{\mu, \mu'\}$ are tangent to λ' at $\{O, O'\}$ and $D \in \lambda'$. This implies that both $\{\lambda, \lambda'\}$ belong to the *"bitangent pencil"* of conics ([2, II, p.187], [9]) and pass both through D, hence they are identical.

Figure 10 illustrates the construction of two triangles corresponding under the affinity. Point $A \in \mu$ can be selected arbitrarily and we take it to be the intersection $A = DJ \cap OO'$. Then, by the method of theorem 5 we construct a secant DBC such that $DB/BC = DA/AJ$. The affinity is the one mapping the triangle OAJ to $O'BC$. □

Two triangles $\{OAJ\ ,\ O'BC\}$, whose vertices correspond under f and lie on the lines $\{\mu, \mu', \mu''\}$, like those of theorem 7 or figure 10, are called *"adapted to the conic of intersection"*.

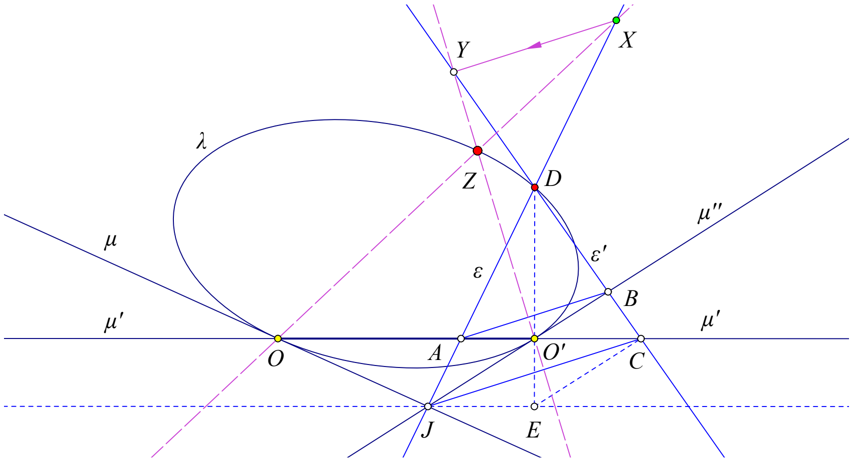

Figure 11: Determination of a conic through parallels

Remark 6. The determination of the affinity producing a given conic as its conic of intersections allows the description of this conic by a simple procedure which uses paralles to a fixed direction. Figure 11 illustrates the case. In this, starting with the points $\{O, O', D\}$ of the conic and its tangents $\{JO, JO'\}$ we constructed the lines $\{\varepsilon = DJ, \varepsilon' = DB\}$ corresponding under the affinity and passing through its fixed point D. Then, drawing from an arbitrary point $X \in \varepsilon$ the parallel $XY, Y \in \varepsilon'$ to AB we obtain the line pair (XO, YO') intersecting at a point Z of the conic.

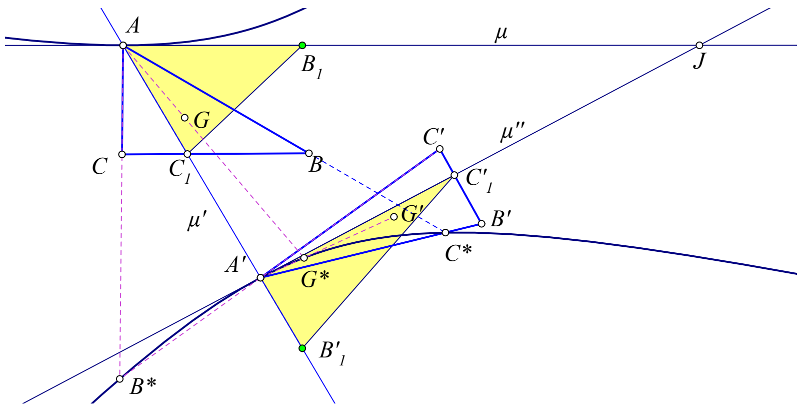

Figure 12: Switching from corresponding triangles to two *"adapted"*

Remark 7. Usually the affinity f is defined by two triangles $\{ABC\ ,\ A'B'C'\}$ supposed to have vertices corresponding under f. In such a case we can switch, if we need, to two *adapted triangles* of the kind used in theorem 7. In fact, starting from the triangles $\{ABC\ ,\ A'B'C'\}$ the conic of intersections λ can be easily constructed using corresponding via f lines $\{\nu = AX, \nu' = A'X'\}$. This conic passes through points $\{A, A'\}$, through the intersection points $\{B^* = AC \cap A'C'\ ,\ C^* = AB \cap A'B'\}$ and through the intersection $G^* = AG \cap A'G'$, where $\{G, G'\}$ the centroids of the triangles $\{ABC, A'B'C'\}$ (see Figure 12). From our discussion follows that the conic λ is tangent to the lines $\mu = f^{-1}(\mu')$

and $\mu'' = f(\mu')$, where $\mu = AA'$. The *adapted* triangles $\{AB_1C_1\ ,\ A'B_1'C_1'\}$ can be defined by selecting two points $\{B_1, C_1\}$ respectively on $\{\mu, \mu'\}$ and taking their images $\{B_1', C_1'\}$ on $\{\mu', \mu''\}$. We notice that AB_1C_1 can be selected to have the same orientation as triangle ABC. Then, also the triangles $\{A'B'C'\ ,\ A'B_1'C_1'\}$ have the same orientation. Thus, the *adapted* triangles can be selected so that their orientation coincides with the one imposed by f. In figure 12 we take $C_1 = \mu' \cap BC$ and B_1 is arbitrary on μ but so that the triangles $\{ABC, AB_1C_1\}$ have the same orientation.

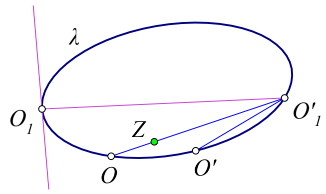

Figure 13: Different pairs $\{(O, O')\}$ define different affinities

Remark 8. Different triples (O, O', D) on a given conic λ produce different affinities having λ as conic of intersections. This is clear for different fixed points $D \in \lambda$. To see this for the pairs $\{(O, O')\}$ consider a conic λ and two different pairs $\{(O, O'), (O_1, O_1')\}$ of points of λ (see Figure 13). Suppose the pairs $\{(O, O'), (O_1, O_1')\}$ define the same affinity f. Then, point O_1' will be the image $O_1' = f(Z)$ of some point $Z \in OO_1'$ but, by assumption, it is also the image of O_1, implying that $O = O_1$. Analogously, considering f^{-1}, is seen that $O_1' = O'$.

8 The image of the conic of intersections

It is easily seen that the image $\lambda' = f(\lambda)$ of the conic of intersections of the affinity f is another conic of intersections of f. In fact, if λ is defined by the intersections of the lines $\{\nu, \nu' = f(\nu)\}$ when ν revolves about the point O, the conic of intersections λ' is created by the intersections of the lines $\{\nu', \nu'' = f(\nu')\}$, the line ν' revolving about $O' = f(O)$ and the corresponding image-line ν'' passing through $O'' = f(O')$ (see Figure 14).

By theorem 4 each one of the conics $\{\lambda, \lambda'\}$ is a dilatation of the other or the conjugate of the other and, since in the generation of the two conics participates the same line ν' carrying both intersection points $\{I, I'\}$ we have the following properties.

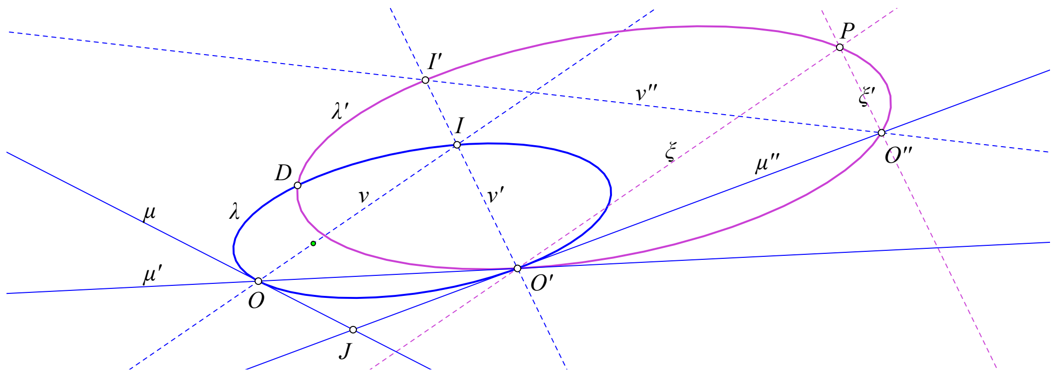

Figure 14: The image $\lambda' = f(\lambda)$ of a conic of intersections is also a conic of intersections

Theorem 8. *With the notation of this section, and assuming that the conic of intersections λ of the affinity f is non-degenerate, the following are valid properties:*

1. *The lines $\nu' = f(\nu)$ through O' intersect the conics $\{\lambda, \lambda'\}$ a second time in corresponding points $\{I, I' = f(I)\}$.*
2. *The point D is a fixed point of f, if and only if it belongs to the intersection $D \in \lambda \cap \lambda'$.*
3. *The tangent of λ' at O' coincides with line $\mu' = OO'$.*
4. *Besides O' the conics $\{\lambda, \lambda'\}$ may have at most one additional intersection point $\{D.\}$*
5. *Affinities whose conic of intersections is an ellipse, have allways an isolated fixed point coinciding with the other than O' intersection point of the conics $\{\lambda, \lambda'\}$.*
6. *Drawing from $\{O', O'' = f(O')\}$ respectively parallels $\{\xi, \xi'\}$ to $\{\nu, \nu'\}$ we obtain a second description of the conic λ' through the intersections $P = \xi \cap \xi'$.*

Proof. *Nr-1* is the comment introducing the theorem.

Nr-2. Every conic of intersections passes through the fixed point D. Conversely if the conics $\{\lambda, \lambda'\}$ intersect a second time at a point D, then the points $\{I, I' = f(I)\}$ on the corresponding line ν' coincide.

Nr-3 follows immediately from the fact, that $\mu' = OO'$ is the image $\mu' = f(\mu)$ and, by theorem 2 μ is tangent to λ at O.

Nr-4. If $\{\lambda, \lambda'\}$ had two intersection points $\{D, D'\}$, then the whole line $\sigma = DD'$ would consist of fixed points, the affinity would be an *axial* one and its conic of intersections would be degenerate, contrary to the hypothesis.

Nr-5. Assume λ is an ellipse not intersecting λ' at an other point $D \neq O'$, then the two conics must be tangent at O' and $\{\mu', \mu''\}$ must coincide, hence their pre-images $\{\mu, \mu'\}$ must also coincide. This means that λ would be tangent to the same line μ' at two different points $\{O, O'\}$, which is impossible.

Nr-6. If ξ is parallel to ν, then, by the preservation of parallels by affinities, $\nu' = f(\nu)$ is parallel to $\nu' = f(\nu)$ and the intersection point $P = \xi \cap \xi'$ is a point of λ'. □

The following example, formulated as a corollary, represents a kind of converse to the circle example of section 6 and the subsequent theorems express some relations between the affinity and its conic of intersections.

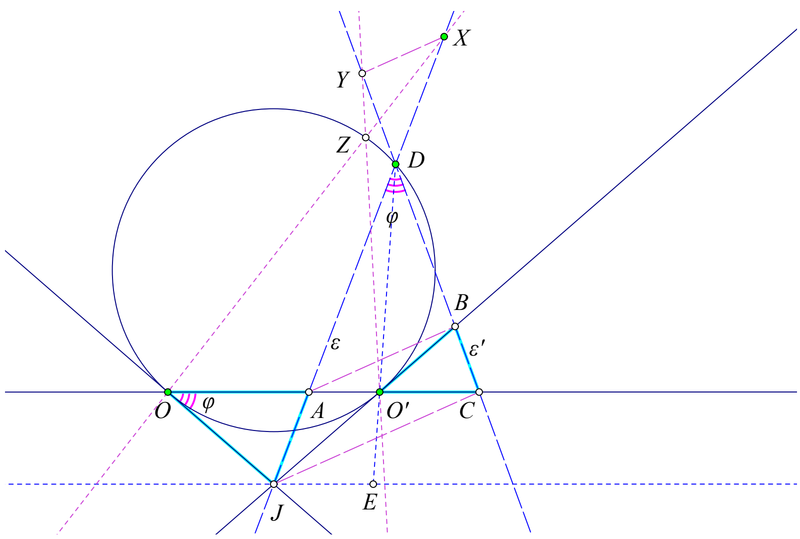

Figure 15: Direct similarities: the conic of intersections is a circle

Corollary 4. *If the conic λ of intersections of the affinity f is a circle, then the affinity is a spiral similarity, i.e. a composition of a rotation $D(\phi)$ and a homothety with the same center D.*

Proof. We use figure 15, which is a special case of the one used in remark 6. By theorem 8(5), the affinity has a fixed point $D \in \lambda$. The angles $\widehat{JOA} = \widehat{CO'B}$ and $\widehat{O'OD} = \widehat{BO'D}$. The ratios $AD/AJ = BD/BC$ are also equal per construction. This implies that the triangles $\{AOD, BO'D\}$ are similar and the triangles $\{JOA, CO'B\}$ are also similar. Thus, f coincides at the vertices of the triangle JOA with the similarity f' centered at D and mapping triangle JOA to its similar $CO'B$. Hence f coincides everywhere with f'. □

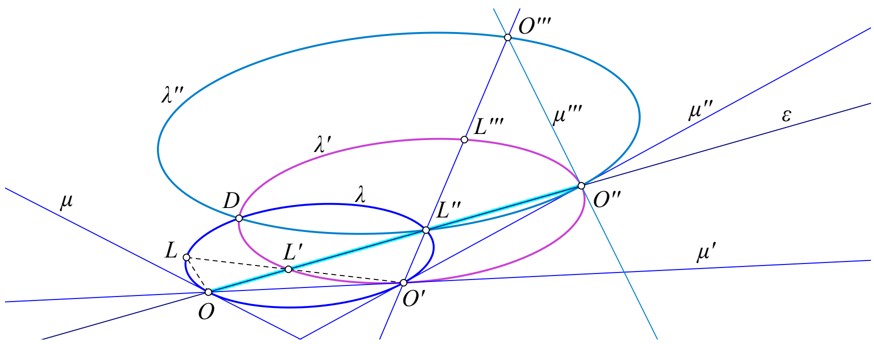

Figure 16: A point $L \in \lambda$ with $L'' = f^2(L) \in \lambda$

Theorem 9. *There is a point L on the conic of intersections λ of the affinity f, such that $f^2(L)$ belongs also to λ.*

Proof. Figure 16 illustrates the case. The prime on a label X' means that $X' = f(X)$. The conic λ is generated by the intersections of lines revolving about O and their images revolving about O'. Analogously are defined the conics $\lambda' = f(\lambda)$ and $\lambda'' = f(\lambda)$ from lines revolving respectively about O' and O'' and their images. We consider the line $\varepsilon = OO''$ intersecting a second time the conic λ' at L'. If L the second intersection of $O'L'$ with λ, we have, according to theorem 8(1), $f(L) = L'$. For the same reason $L'' = f(L') \in OO''$ and since $L' \in OO'' \Rightarrow L'' \in O'O''' \Rightarrow L'' = OO'' \cap O'O'''$. By the definition of λ, since $O'O''' = f(OO'')$ point $L'' \in \lambda$. □

Theorem 10. *Consider an affinity f, a point O with $O' = f(O) \neq O$ and the conic of intersections λ defined by (O, O'). If these two points are on the same component of λ, then f preserves the orientation and if they are on different components, then f reverses the orientation.*

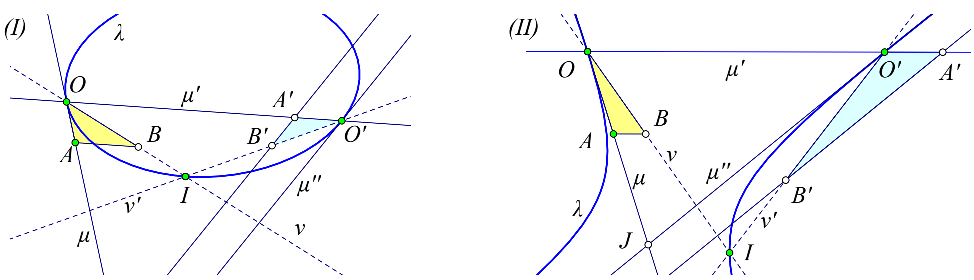

Figure 17: The behavior of orientation under f

Proof. Consider first the case of a pair (O, O') belonging to the same component and take an arbitrary point $I \in \lambda$ together with the lines $\{\nu = OI, \nu' = O'I = f(\nu)\}$ (see figure 17-(I)). Then take an arbitrary point $A \in \mu$ and $B \in \nu$ such that AB is parallel to μ'. The image-triangle $A'B'O'$ of ABO will have the side $A'B'$ parallel to μ'' and we can easily

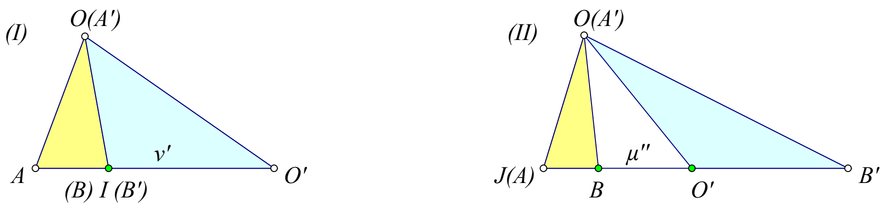

Figure 18: Detecting the orientations of $\{\triangle OAB\ ,\ \triangle O'A'B'\}$

see that the two triangles have the same orientation. In fact, deforming continuously the two triangles without to change their orientation, we can bring them in the configuration of figure 18-(I), showing that the triangles are equally oriented. In the case points (O, O') belong to different components, analogous work illustrated by figure 17-(II) and subsequent deformation illustrated by figure 18-(II) shows that the two triangles are oppositely oriented. □

Corollary 5. *Affinities whose intersection conic is an ellipse or a parabola preserve the orientation. This is true also for affinities whose intersection conic is a hyperbola, but the pair of points (O, O') generating it belongs to the same component. If the intersection conic is a hyperbola and the generating points $\{(O, O')\}$ belong to different components, then the affinity reverses the orientation.*

Theorem 11. *Every conic λ is the conic of intersections of an equiaffinity.*

Proof. Given the conic λ, it suffices to find an adapted pair of triangles of equal areas and equal orientations. Figure 19 illustrates this in the case of an ellipse. In this we

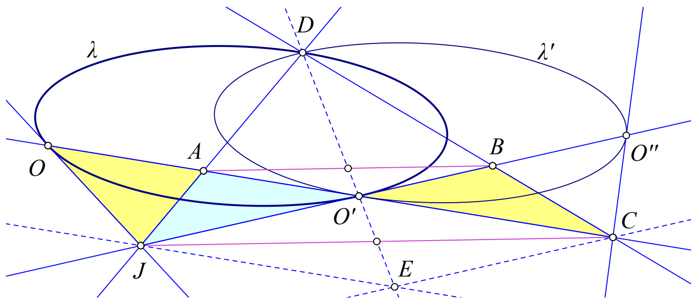

Figure 19: Arbitrary conic λ generated by an appropriate equiaffinity

consider two arbitrary points $\{O, O'\}$ of λ and take $D \in \lambda$ on the conjugate diameter of $\mu' = OO'$ passing through the middle A of μ. Using the method of theorem 7 we construct the triangle OBC. By the discussion in this section we know that $O'' = f(O')$ is on line JO' and $B = f(A)$ is the middle of $O'O'' = f(OO')$, since A is the middle of OO'. Since $\{AB, JC\}$ are parallels the triangles $\{ABJ, ABC\}$ have equal areas and both contain the triangle $AO'B$. This implies the equality of the areas of triangles $\{O'BC, O'AJ\}$ latter having the same area with $\triangle\ OJA$. The equiaffinity satisfying the requirements, as usual is defined by the correspondence of vertices of triangles $\{AOJ, BO'C\}$.

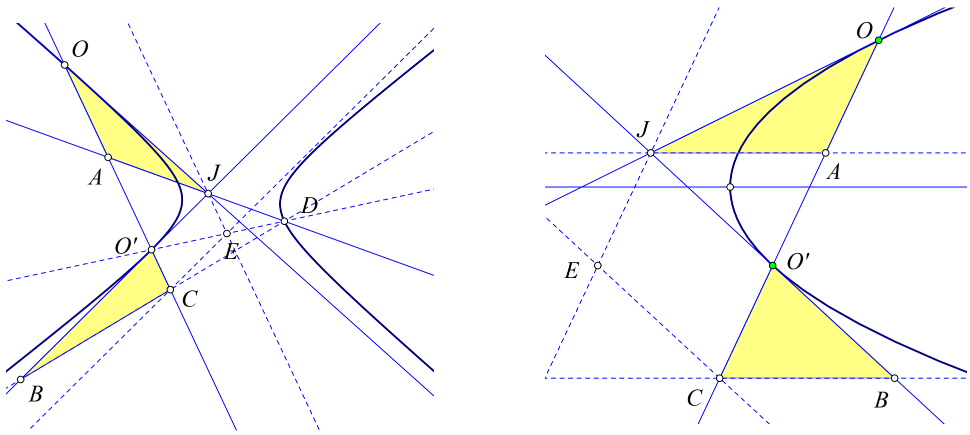

Figure 20: Arbitrary hyperbola/parabola generated by an appropriate equiaffinity

Figure 20 illustrates the corresponding construction in the case of hyperbolas and parabolas. In the hyperbola the points $\{O, O'\}$ are taken on one branch and D on the other. In the case of the parabola lines $\{AJ, O'E\}$ are parallel to the axis. Also in these cases it is easily verified that the triangles $\{AOJ, BO'C\}$ of the adapted pair satisfy the requirements. □

9 Affinities without fixed points related to hyperbolas

Here we complement the discussion initiated in section 7 examining the possibility to define an affinity f *without fixed points* having a given non degenerate conic λ as its conic of intersections. By theorem 8(5) the conic λ cannot be an ellipse. Thus, we seek to define an affinity without fixed points, having corresponding conic of intersections a hyperbola or a parabola. Here we examine the case of hyperbolas.

By the general properties analyzed in theorem 2, the affinity f induces in its conic of intersections λ a natural structure consisting of two points $\{O, O'\}$ in λ, the tangents $\{\mu, \mu''\}$ there and the line $\mu' = OO'$. Thus, we can start from an arbitrary such structure on the given conic and seek to define a corresponding affinity. In the case of hyperbola a crucial role in our study plays the following property (see Figure 21).

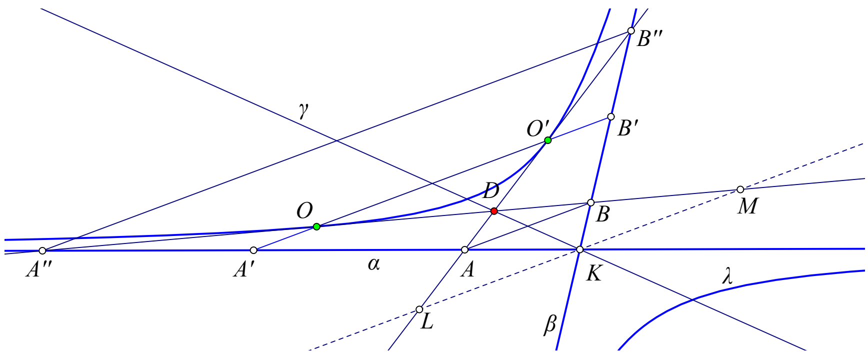

Figure 21: A property of the hyperbola

Lemma 2. *Let $\{\alpha, \beta\}$ be the asymptotes of the hyperbola λ. Let $\{(A'', B), (A, B'')\}$ be the intersections of the asymptotes with the tangents at the points $\{O, O'\}$ and $\{A', B'\}$ be the intersections* *of the asymptotes with the line* OO'. *Then* $\{AB, A'B', A''B''\}$ *are parallel lines and* $A'B'$ *is the middle-parallel of the other two.*

Proof. Consider the intersection $D = A''B \cap AB''$ and the line $\gamma = KD$, point K being the center of the hyperbola. The diagonals $\{AB, A''B''\}$ of the quadrilateral $AKBD$ and the sides $\{AB, A''B''\}$ of the quadrilateral $AB''A''A'$ intersect on the harmonic conjugate line γ' of γ w.r.t. $\{\alpha, \beta\}$. Hence, the three lines $\{AB, A'B', A''B''\}$ pass through a point of γ'. The line from this point to K intersects the tangents $\{AB'', A''B\}$ at the points $\{L, M\}$ and the cross ratios are equal:

$$(A''B; OM) = \frac{OA''}{OB} : \frac{MA''}{MB} = (B''A; O'L) = \frac{O'B''}{O'A} : \frac{LB''}{LA} \quad \Rightarrow \quad \frac{MA''}{MB} = \frac{LB''}{LA}, \tag{11}$$

since, by the well known property of hyperbolas $OA''/OB = O'B''/O'A = -1$. The fact that $\{LM, AB, A''B''\}$ pass through the same point and the equality 11 imply that these lines are parallel. This follows easily from the equality of cross ratios $(AD; OM) = (B''D; O'L)$ and leads to the proof of the lemma. □

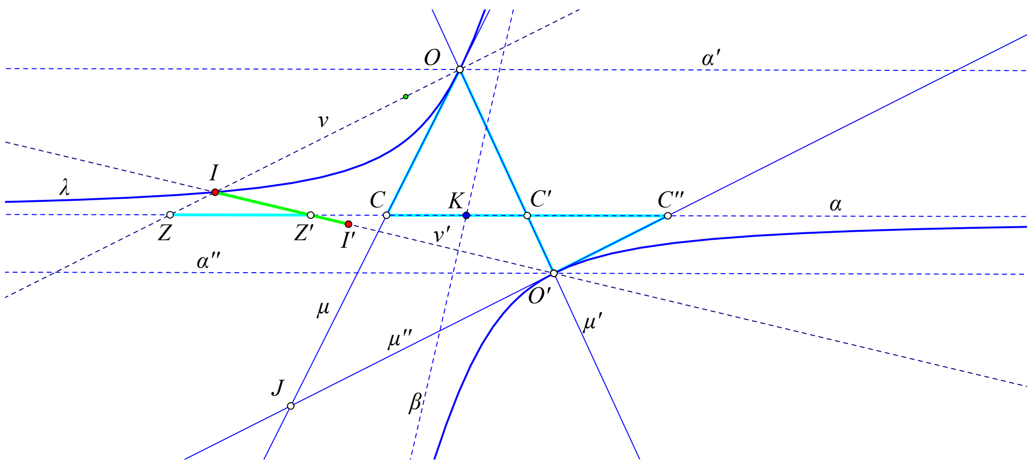

Figure 22: Defining an affinity without fixed points

Figure 22 illustrates the procedure of definition of an affinity f without fixed points having a given hyperbola as its conic of intersections. The starting point are two points $\{O, O'\}$ on λ and the tangents there $\{\mu, \mu''\}$. By lemma 2, the intersection points $C = \mu \cap \alpha$, $C' = \mu' \cap \alpha, C'' = \mu'' \cap \alpha$, where α an asymptotic line of λ and $\mu' = OO'$, define segments $CC' = C'C''$. We define f as the affinity mapping the vertices of triangle OCC' to corresponding vertices of the triangle $O'C'C''$. It follows that $f(\mu) = \mu'$, $f(\mu') = \mu''$ and $f(\alpha) = \alpha$ i.e. line α remains invariant under f. Denoting by λ^+ the conic of intersections of f, we see easily that λ^+ has $\{\mu, \mu''\}$ as tangents at O, O' and that it is a hyperbola having α as asymptote, mapping a point $Z \in \alpha$ to Z' with $ZZ' = CC'$. The following easily to prove properties of λ^+ lead to its identification with λ (see Figure 23).

1. The equality $ZZ' = CC'$, taking in particular Z such that ZO is parallel to the other asymptote β of λ makes $Z'O'$ parallel to ZO showing that the other asymptote of λ^+ is also parallel to β.
2. This implies that a line ε parallel to β is shifted by the vector $v = ZZ' = CC'$ to a parallel ε'.
3. The parallel α' to α from O maps to the parallel α'' from O'. More general a parallel γ to α maps to a parallel γ' to α and if $X = \mu' \cap \gamma$ and $Y = \mu' \cap \gamma'$, then $XC'/YC' = OC'/O'C' = k$ is constant.

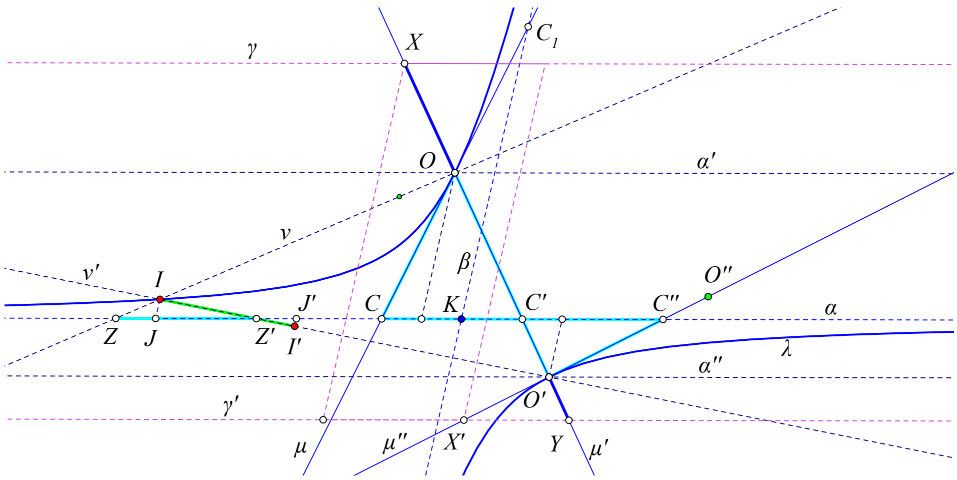

Figure 23: Identifying λ with λ^+

4. If for a point $I = \nu \cap \nu'$ of λ^+ the image point $I' = f(I) \in \nu'$, where $\{\nu, \nu' = f(\nu)\}$ are lines through $\{O, O'\}$ generating the hyperbola λ^+ and $Z = \nu \cap \alpha, Z' = f(Z) \in \nu'$, then $IZ'/I'Z' = OC'/O'C' = k$.
5. If $\{J, J'\}$ are the projections of $\{I, I'\}$ parallel to β on α, then $JJ' = ZZ' = CC'$ and $ZJ = Z'J'$.
6. When I takes the position O, the corresponding Z obtains the position of C and $IZ = \mu$ is the tangent to both λ and λ^+. Then, the symmetric of $Z = C$ w.r.t. O projects parallel to β on the center K of the hyperbola λ. As the other than α asymptote of λ^+ is parallel to β the preceding property implies that K is also the center of λ^+.
7. Since the two conics $\{\lambda, \lambda^+\}$ have at $\{O, O'\}$ the same tangents and also have the same center, they are identical.

Since the projection JJ' of II' on α parallel to β has constant length $JJ' = CC'$ we have $I \neq I'$ for all positions of ν and the affinity f has no fixed point. Introducing coordinates along the asymptotes of λ, the representation of f is:

$$\begin{aligned} x' &= x + v_1, \\ y' &= k \cdot y, \end{aligned}$$

where $v_1 = CC'$ and $k = OC'/O'C'$ is a constant ratio of signed lengths. This represents a generalization of the affine glide reflection, which is a composition $f = h \circ g$ with g a strain with axis α and ratio k and h is a translation by a vector $(v = (v_1, 0))$ parallel to the axis of the strain. We formulate all that as a theorem.

Theorem 12. *Any hyperbola is the conic of intersections of an affinity f without fixed points, represented as a composition $f = h \circ g$ of a strain g and a translation h parallel to the axis of the strain.*

10 Affinities without fixed points related to parabolas

Figure 24 illustrates the construction of an affinity f without fixed points having a given parabola λ as its conic of intersections. In this we start with two arbitrary points $\{O, O'\}$ on the parabola and the tangents $\{\mu, \mu'\}$ at these points intersecting at the point J. The triangle OAJ is defined by the intersection of line $\mu' = OO'$ and the parallel to the axis ε of the parabola. The side BC of the triangle $O'BC$ is the symmetric of JA w.r.t. O'. The affinity f is defined by the correspondences $\{O \mapsto O'$, $J \mapsto C$, $A \mapsto B.\}$ By theorem 2 the conic λ^+ of intersections I of f passes through $\{O, O'\}$ and is tangent there to the lines $\{\mu, \mu''\}$. The affinity maps the middle M of AJ to the middle M' of CB and line MM' passes through O', implying that $M = OM \cap O'M'$ is a point of λ^+. Thus, the two conics $\{\lambda, \lambda^+\}$ belong to the bitangent pencil $\{s \cdot \mu\mu'' + t \cdot \mu'^2, s, t \in \mathbb{R}\}$ and pass both through M, hence they are identical. It is easily seen that a line α parallel to the axis ε of the parabola maps to a parallel α'. Finally the distance d of the two parallels $\{\alpha, \alpha'\}$ is constant.

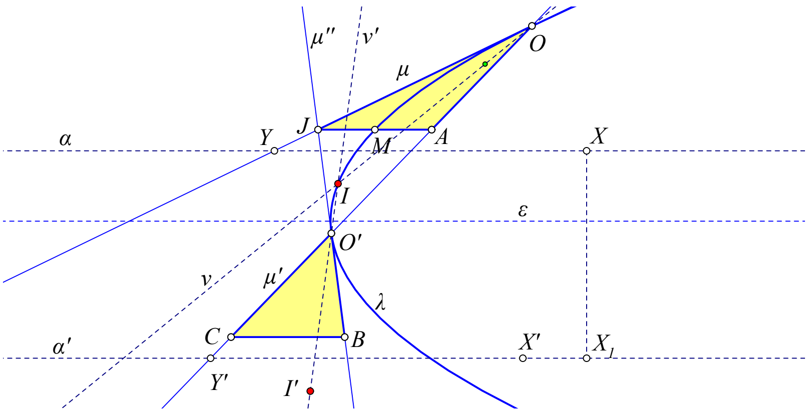

Figure 24: Affinity without fixed points related to a given parabola

In fact, if $\{Y = \alpha \cap \mu$, $Y' = f(Y) = \alpha' \cap \mu'\}$, then $OJ/JY = O'C/CY'$ and the fact that A is the middle of OO' for a parabola, implies that the distance of the parallels $\{JA, \alpha\}$ is equal to the distance of their images $\{CB, \alpha'\}$. This implies that d is equal to the distance of the parallels $\{JA, CB\}$ which is constant. A consequence of this is that the parallels to ε through the points $\{I, I' = f(I)\}$ are at a distance d, hence the points $\{I, I'\}$ are never coincident and f has no fixed points.

Taking point O' as origin of a cartesian coordinate system with axis parallel resp. perpendicular to the axis ε of the parabola we find easily the expression of the affinity in the form:

$$\begin{aligned} x' &= x + k \cdot y + v_1, \\ y' &= \qquad y + v_2, \end{aligned}$$

with constants $\{k, v_1, v_2\}$, representing the composition of a shear and a translation. We formulate these results as a theorem.

Theorem 13. *Any parabola is the conic of intersections of an affinity f without fixed points, represented as a composition $f = h \circ g$ of a shear g and a translation h.*

11 Orbital conics of equiaffinities

As we noticed earlier *"equiaffinities"* build a subgroup of the group of affinities and, per definition, preserve the areas of triangles. This has some implications among which an important one ([3]) is the fact that the *"orbit"* of a point X i.e. the sequence of points $\{X, f(X), f^2(X), \ldots\}$ is contained in one conic. Something that is not valid for a general affinity f. Here we show that the conic containing an orbit of the equiaffinity is a dilatation of the conic of intersections. For this we need some additional properties concerning the relation of a conic of intersections and its image, illustrated in figure 25. In this,

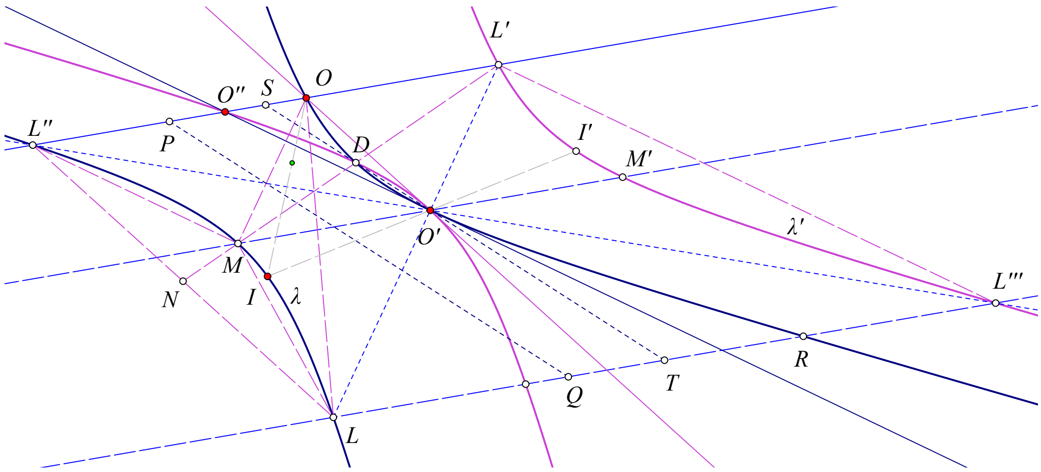

Figure 25: Conic of intersections λ of an equiaffinity and its image $\lambda' = f(\lambda)$

as usual, $\{\lambda, \lambda' = f(\lambda)\}$ denote the conic of intersection and its image under the affinity. Points $\{O, O' = f(O)\}$ are used to generate λ through the intersection points $\{I\}$. As we noticed already, lines through O' intersect a second time the conics λ, λ' at points $\{I, I'\}$ corresponding under the affinity f. In particular point $O'' = f(O')$ of λ' is on the tangent to λ at O'. In the figure points $\{L', L''\}$ are respectively the intersections of the line OO'' with the two conics $\{\lambda', \lambda\}$ and the points $\{L, L'''\}$ are defined by $\{f(L) = L', f(L'') = L'''\}$. Also MO' is parallel to $L'L''$. In the following, saying that *"λ' is a dilatation of λ"*, we mean that λ' is indeed a dilatation of λ or a dilatation of the conjugate λ^* in the case λ is a hyperbola. We make also the following observations:

1. $\triangle\, OO'M$ maps via f to $\triangle\, O'O''M'$. The two triangles have the same area and that of the second is equal to the area of $\triangle\, OO'M'$, implying that $MO' = O'M'$.
2. $\triangle\, MOL$ maps via f to $\triangle\, M'O'L'$ of equal area, later equal to the area of $\triangle\, OMO'$, because $MM' \parallel L'L''$ and $MO' = O'M'$. Thus, triangles $\{MOL\,,\, MOO'\}$ have equal areas and MO is parallel to LL'.
3. It follows that $MO'L'O$ is a parallelogram, $OL' = MO' = O'M'$ and $O'M'L'O$ is also a parallelogram.
4. By theorem 9 $L'' = f(L')$, consequently $\triangle\, OLL'$ maps via f to $\triangle\, O'L'L''$ of equal area, the two triangles sharing the area of $\triangle\, OO'L'$. This implies that the triangles $\{OO'L, OO'L''\}$ have the same area, hence OO' is parallel to LL''.
5. For the same reason triangles $OO'L', O'O''L''$ correspond via f and have equal area, implying that $O''L'' = OL'$.
6. Since $\{LL'', O'O\}$ are parallel chords of λ the middle N of LL'' defines their conjugate direction which is line NL' and passes also through the middle of OO'. Since $O'M'L'O$ is a parallelogram line NL' passes through M.
7. The tangent to λ at M is parallel to $LL'' \parallel OO'$ latter being the tangent of λ' at O'.
8. The tangent to λ at M which is parallel to OO' maps via f to the tangent at M' of λ' which must be parallel to $O'O''$ which is tangent to λ at O'. Also since LL'' and the tangent to λ at M are parallel and map to $L'L'''$ and the tangent to λ' at M', these two lines must be parallel to, implying that $L'L''' \parallel O'O''$.

9. The lines $\{L'L'', MM'\}$ being parallel define a trapezium and $\{L'M, L''M' = f(L'M)\}$ are its diagonals. If they intersect, then, from the properties of trapezia, their intersection point D satisfies $MD/DL' = M'D/DL''$, hence coincides with the fixed point of the affinity.
10. Triangle $LL'L''$ maps via f to triangle $L'L''L'''$ of equal area, hence line LL''' is parallel to $L'L''$.

Theorem 14. *The image $\lambda' = f(\lambda)$ of the conic of intersections λ of an equiaffinity f is a translation of λ.*

Proof. In fact, by the preceding observations, the translation t_v by the oriented segment $v = OL'$ maps the points $\{O, O', M, L''\}$ of λ respectively to $\{L', M', O', O''\}$ and the tangents to λ at $\{M, O'\}$ respectively to the tangents of λ' at $\{O', M'\}$, all this guaranteeing ([10]) that $\lambda' = t_v(\lambda)$. □

Corollary 6. *The line PQ joining the middles $\{P, Q\}$ of the parallel chords $\{L''O, LR\}$ of λ is parallel to the line ST joining the middles of the segments $\{L'L'', LL'''\}$ (see Figure 25).*

Proof. This follows from a trivial property of trapezia, according to which the line joining the middles of its parallel sides has a fixed direction if the sides of the trapezia have also fixed directions. The claim follows from this and the fact, that $\{LL'''L'L'', LROL''\}$ are trapezia having their parallel sides on the same lines and their sides $\{L'L''', OR\}$ parallel, since, by the preceding theorem $RL'''L'O$ is a parallelogram. □

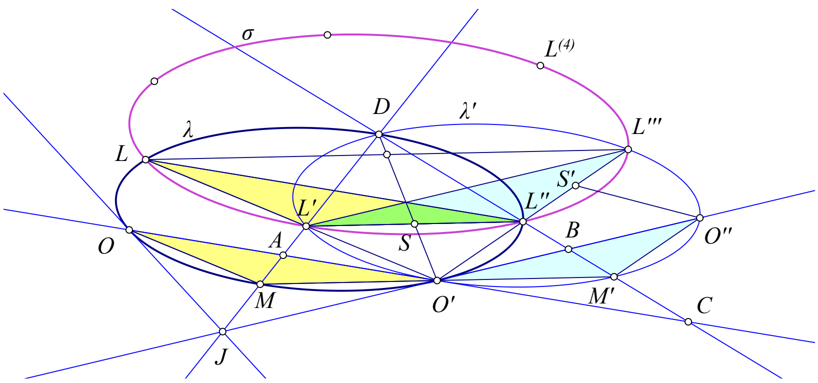

Figure 26: Orbital conic σ is a dilatation of λ

Theorem 15. *A conic σ containing an orbit of an equiaffinity f is a dilatation of a conic of intersections λ of f and is non-degenerate if and only if λ is non-degenerate.*

Proof. We use the point $L \in \lambda$ of theorem 9 with $f^2(L) \in \lambda$ and the observations related to theorem 14. We consider the orbital conic σ containing the sequence of points $\{L, L' = f(L), L'' = f(L'), \ldots\}$ (see Figure 26).

From the aforementioned observations follows that the triangles $\{LL'L'', OMO'\}$ have parallel corresponding sides and the same happens also for triangles $\{O'M'O'', AL''L'''\}$. Besides, from the same observations follows that $\{LL''', L'L''\}$ are parallel chords of σ and, consequently, the line $O'S$ joining their middles is the conjugate direction of $L'L''$ w.r.t. σ. From corollary 6 we have however that the directions of the lines $\{L'L'', O'S\}$ are conjugate also w.r.t. λ. Hence the two conics $\{\lambda, \sigma\}$ have a common pair of conjugate directions.

The construction of the configuration of the four triangles $LL'L''$, OMO', $L'L''L''$, $O'M'O''$ depending on L can be repeated starting with $L' = f(L)$. This leads to a similar configuration of four triangles and a pair of directions $\{L''L''', O''S'\}$ which are conjugate w.r.t. both conics $\{\lambda', \sigma\}$. But, since λ' is a translation of λ, they are common conjugate also to λ, σ.

The result follows from the fact, that two non-degenerate conics with two common pairs of conjugate directions represented in a cartesian coordinate system have proportional quadratic coefficients. In fact, assuming ((a,b)) and (c,d) are pairs of unit vectros expressing two common conjugate directions of the conics $Ax^2 + 2Bxy + Cy^2 + ... = 0$ and $A'x^2 + 2'B'xy + C'y^2 + ... = 0$, the conjugacy condition implies that both triples of coefficients satisfy the equations:

$$\left.\begin{array}{rcl} Aa_1b_1 + B(a_1b_2 + a_2b_1) + Ca_2b_2 &=& 0, \\ Ac_1d_1 + B(c_1d_2 + c_2d_1) + Cc_2d_2 &=& 0. \end{array}\right\} \tag{12}$$

Thus, considering this as a linear system of equations for three-dimensional vectors, the vectors $\{(A,B,C), (A',B',C')\}$ must be both multiples of the exterior product

$$\begin{pmatrix} a_1b_1 \\ a_1b_2 + a_2b_1 \\ a_2b_2 \end{pmatrix} \times \begin{pmatrix} c_1d_1 \\ c_1d_2 + c_2d_1 \\ c_2d_2 \end{pmatrix}.$$

This implies the proportionality of the quadratic terms, on which depend the main characteristics of the conics showing, as in the theorem 4, that each is a dilatation of the other. We notice here that, if the conic λ is non-degenerate, then also the orbital conic is non-degenerate and vice versa. This follows from the similarity of the triangles $\{OMO', LL'L''\}$ which are respectively inscribed in the two conics. □

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