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TRIANGLES SHARING THEIR EULER CIRCLE AND CIRCUMCIRCLE

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Abstract. In this article we study properties of triangles with given circumcircle and Euler circle. They constitute a one-parameter family of which we determine the triangles of maximal area/perimeter. We investigate in particular the case of acute-angled triangles and their relation to poristic families of triangles. This relation is described by an appropriate homography, whose properties are also discussed.

1 Introduction

The scope of this article is to explore properties of triangles sharing their *"Euler circle"* $\kappa(N, r)$ and their circumcircle $\kappa'(O, 2r)$. All these triangles for fixed $\{\kappa, \kappa'\}$ are called *"admissible"*, and as we'll see shortly, they constitute a one-parameter family, which we call an *"admissible family of triangles"*. By well known theorems, for which a good reference is the book by Court [3], the Euler circle has its center N on the *"Euler line"* ε midway between the circumcenter O and the orthocenter H of the triangle ABC (See Figure 1).

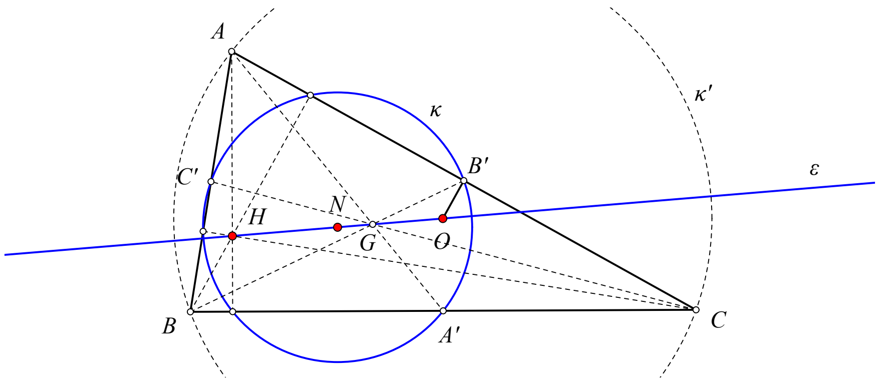

Figure 1: Euler line and circle

In principle, the location of these two points, which lie symmetrically w.r. to N, can be arbitrary within certain bounds to be noticed below. Assuming the Euler circle to be con-

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tained totally inside the circumcircle $\kappa'(O, 2r)$ we can define corresponding admissible triangles as follows.

We select an arbitrary point B' on κ and draw the line AC orthogonal to OB' intersecting κ' at $\{A, C\}$. A third point B and a triangle ABC defining an *admissible triangle* is found using the centroid G of the triangle, lying also on line ε, its position dividing the segment ON in ratio $OG/GN = 2$. The location of the third point B is on the intersection $B'G \cap \kappa'$.

The proof that the created triangle ABC is *admissible*, is an easy exercise, which can start by observing the similarity of the triangles $\{GOB', GHB\}$ which are (anti)homothetic w.r. to G and in ratio $1/2$. This homothety, or equivalently the parallelity of BH to OB', is also the criterion of choosing between B and its antipode w.r. to κ'.

The shape of the thus created admissible triangles depends on the relation of the radius r to the distance $h = |HN| = |NO|$. For $r > h$ all these triangles are acute-angled as in figure 1. For $r = h$ the triangles are right-angled as in figure 2, in which $\{B', H\}$ are respectively identical with $\{O, B\}$ and the Euler line is the median BB' of the triangle. Figure 3 shows the case of $r < h$ for which all admissible triangles are obtuse. The

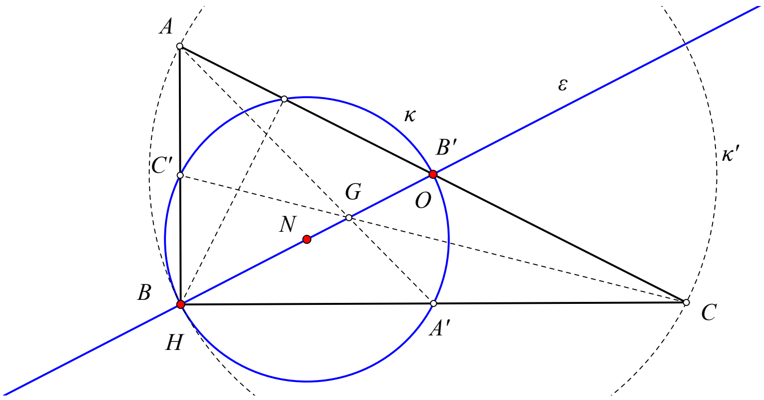

Figure 2: The case of right-angled triangles

possible locations of B' extend not to the whole of κ but only to the arc of the circle contained inside the circumcircle κ'. If B' obtains the position of one of the extremities $\{P, Q\}$ of this arc, then the corresponding line orthogonal to OB' at B' is tangent to κ' and no genuine admissible triangle ABC can be defined. The chords $\{PGP', QGQ'\}$ of κ' intersect κ at two points $\{P_0, Q_0\}$ which are positions of B' for which the correspondingly triangle degenerates correspondingly to the segments $\{PP', QQ'\}$. The case $h \geq 3r$ is not possible, since then for all positions of $B' \in \kappa$ no genuine admissible triangle can be defined. Thus, given the position of N on line ε, the point H or equivalently O has to be selected at a distance from N less than $3r$.

2 Area and perimeter variation

Given the Euler circle κ and the circumcircle κ' and considering, for the time being, the case of acute-angled triangles, we can study the variation of the area and perimeter of the admissible triangles in dependence of the variable point $B' \in \kappa$. First we should observe that, because of the symmetry of the figure w.r. to line ε (See Figure 4), the different shapes of the admissible triangles are all obtained for the positions of B' within the arc B_1B_2 of κ. The two extremities $\{B_1, B_2\}$ of this arc are precisely the midpoints of the corresponding sides $\{AC\}$ of the two unique isosceli admissible triangles $\{\tau_1,\ \tau_2\}$ which

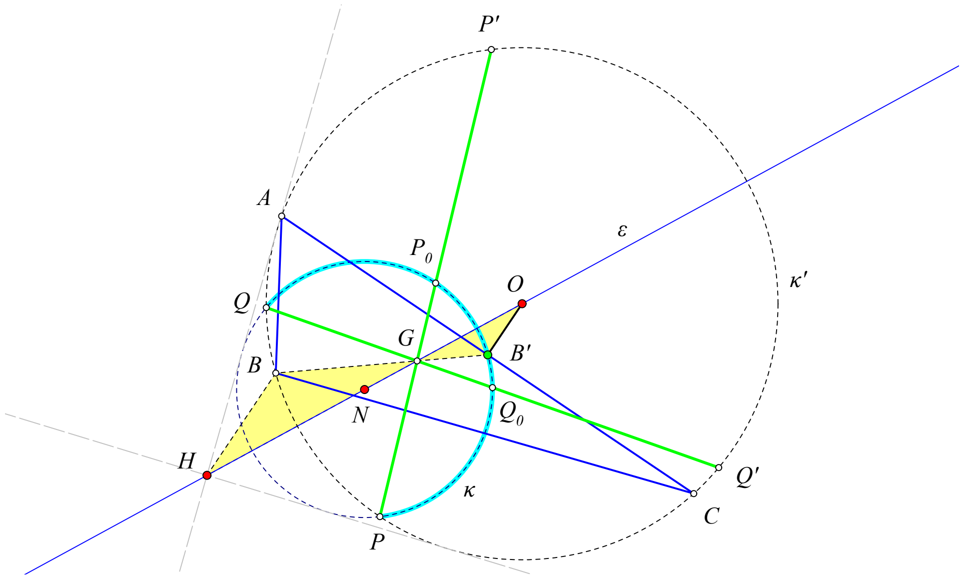

Figure 3: The case of obtuse-agled triangles

are symmetric w.r. to ε. A similar discussion for the circumcircle and the incircle can be found in [1].

Theorem 1. *With the notation and conventions introduced so far, the two triangles* $\{\tau_1, \tau_2\}$ *possess the maximal resp. minimal area and perimeter among the admissible ones.*

Proof. By the preceding remark, to find the maximal/minimal area/perimeter of the admissible triangles it suffices to examine those for which the middle B' of their side AC is on the indicated arc B_1B_2. Thus, it suffices to show that the area and the perimeter are decreasing as B' moves on this arc from B_1 towards B_2.

For the area this can be seen by expressing it in terms of $b = |AC|$ and the altitude $u = |BB''|$, as a function $f(x)$ of $x = |OB'|$, varying in the interval corresponding to the arc B_1B_2. Latter is readily seen to be

$$r - h \;\leq\; x \;\leq\; \sqrt{r(r-h)}\,. \tag{1}$$

Considering positive areas and using $f^2(x)$ instead, we find the expressions:

$$b^2 \;=\; 4(4r^2 - x^2), \quad |BB''| \;=\; 2|OB'| + |HB''| \;=\; 2x + y \quad \text{with} \quad xy \;=\; r^2 - h^2 \quad \Rightarrow$$
$$|BB''| = \frac{2x^2 + r^2 - h^2}{x} \quad \Rightarrow \quad f^2(x) \;=\; \frac{(4r^2 - x^2)(2x^2 + r^2 - h^2)^2}{x^2}.$$

Doing a bit of calculus we see that this is indeed strictly decreasing in the interval (1).

Analogously, the decreasing behavior of the perimeter can be seen by expressing it in terms of $\{u = |BB''|,\ b = |AC|,\ v = |B'B''|\}$ and doing some computation:

$$v^2 \;=\; (2h)^2 - (y - x)^2 \quad \Rightarrow$$
$$a^2 \;=\; u^2 + (b/2 + v)^2, \qquad c^2 \;=\; u^2 + (b/2 - v)^2 \quad \Rightarrow$$
$$a^2 + c^2 \;=\; 2u^2 + b^2/2 + 2v^2 \;=\; 4(x^2 + 5r^2 - h^2)\,,$$
$$a \cdot c \;=\; \frac{4r(2x^2 + r^2 - h^2)}{x} \quad \Rightarrow \quad (a+c)^2 \;=\; \frac{4((x+r)^2 - h^2)(x + 2r)}{x} \quad \Rightarrow$$
$$a + b + c \;=\; 2\sqrt{2r + x}\left(\sqrt{2r - x} + \sqrt{\frac{(x+r)^2 - h^2}{x}}\right).$$

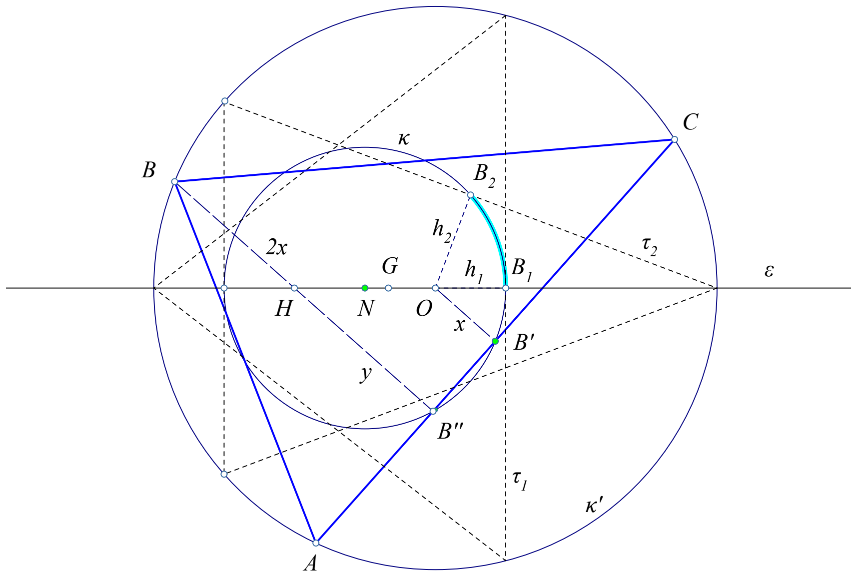

Figure 4: Calculating the area

Doing again a bit of calculus we see that the function is decreasing in the interval (1). Figure 5 gives an impression of the behavior of this perimeter function, modified by a

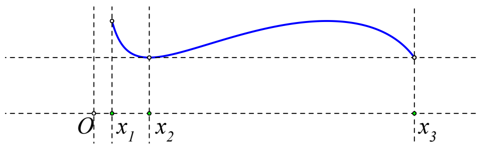

Figure 5: The perimeter function

small multiplicative factor, in the interval $x_1 = r - h \leq x \leq r + h = x_3$, with $x_2 = \sqrt{r(r-h)}$ defining the right end of the interval (1). $\square$

Corollary 1. *The area* (ABC) *resp. perimeter* 2τ *of the triangle* ABC *with given Euler circle* $\kappa(N,r)$ *and distance from the orthocenter* $|EN| = h < r$ *is bounded by*

$$(3r+h)\sqrt{4r^2-(r+h)^2} \leq (ABC) \leq (3r-h)\sqrt{4r^2-(r-h)^2}\,, \tag{2}$$

$$2(\sqrt{r-h}+2\sqrt{r})\sqrt{3r+h} \leq 2\tau \leq 2(\sqrt{r+h}+2\sqrt{r})\sqrt{3r-h}\,. \tag{3}$$

Remark 1. On the ground of a geometric definition of the ellipse, it can be proved and is well known ([17], [18, p.131]) that all admissible triangles have the following property, which we formulate as a theorem, giving also another aspect of the pair of points $\{O,H\}$.

Theorem 2. *All triangles sharing the same Euler circle* $\kappa(N,r)$ *and circumcircle* $\kappa'(O,2r)$, *and satisfying* $h = |HN| = |ON| < r$ *have their sides tangent to the ellipse with focals at* $\{H,O\}$ *and eccentricity* h/r. *Their corresponding side-lines are the orthogonal bisectors of the segments* HP, *where* P *is a point varying on the circumcircle* κ' *(See Figure 6).*

3 Obtuse and right-angled trianlges

Continuing with the notation and conventions of the preceding sections, the case of *obtuse* admissible triangles, characterized by the condition $r < h$, is illustrated by figure 7. In

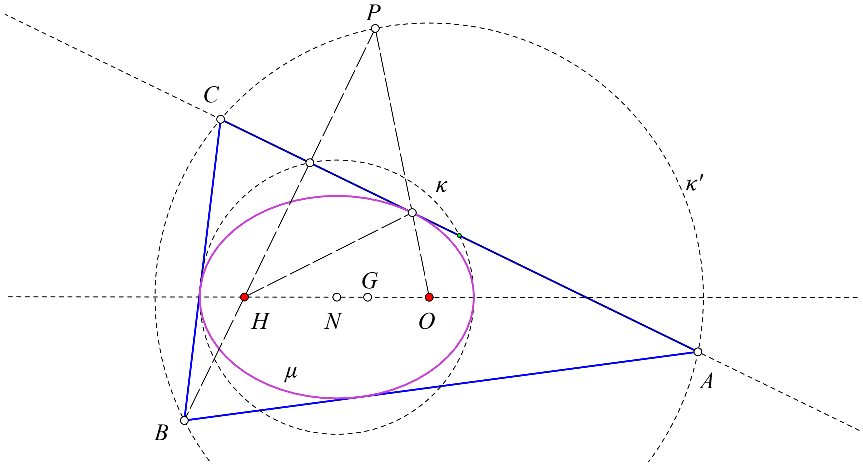

Figure 6: Enveloping ellipse of side-lines of admissible acute triangles

this case the side-lines of the admissible triangles envelope a hyperbola with *"auxiliary circle"* κ and *"circular-directrix"* κ'. By studying the variation of the chord AC, we see easily that the measure of the obtuse angle ω varies between two values

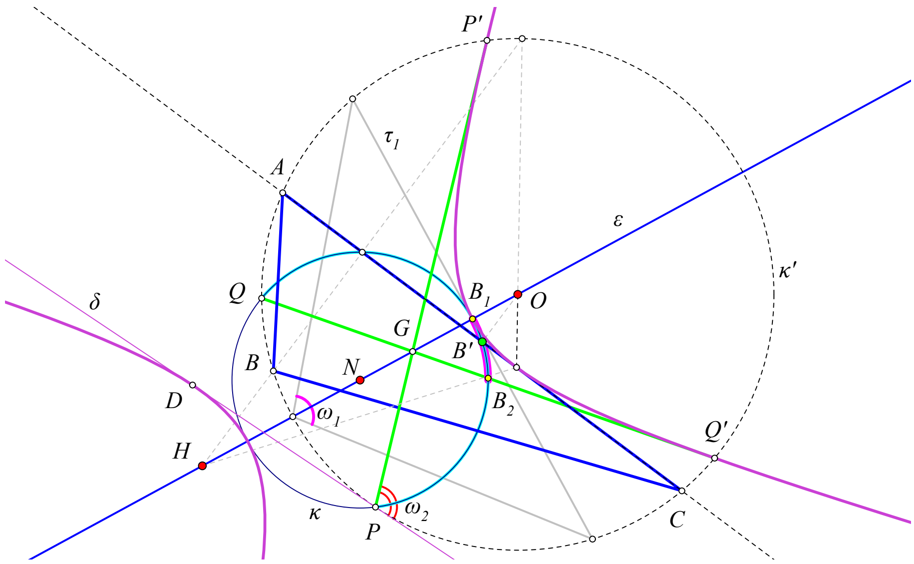

Figure 7: Enveloping hyperbola of side-lines of admissible obtuse triangles

$$\omega_1 \ \leq \ \omega \ \leq \ \omega_2 \ . \tag{4}$$

The smaller ω_1 is the apex-angle of the maximal isosceles τ_1 and the bigger one is the angle ω_2 to which tends the measure of the obtuse angle, as the triangle tends to degeneration to the (double) segment PP'. Notice that in this case the vertex A of the variable triangle ABC tends to P' and the side-line BC tends to the tangent δ of κ' at P. Since this side-line is all the time tangent to the hyperbola, δ is also tangent to it. Analogous to the inequalities (4) are satisfied by the measures of the angles also in the case of acute-angled triangles. In this case $\{\omega_1, \omega_2\}$ are correspondingly the apex-angles of the minimal resp. maximal isosceli. Next theorem formulates the analogous to theorem 2 for obtuse-angled triangles.

Theorem 3. *All triangles sharing the same Euler circle $\kappa(N,r)$ and circumcircle $\kappa'(O)$, and having $h = |HN| > r$ have their sides tangent to the hyperbola with focals at $\{H,O\}$ and eccentricity h/r. Their corresponding side-lines are the orthogonal bisectors of the segments HP, where P is a point varying on the circumcircle κ' (See Figure 7).*

The necessary computations for the area and the perimeter of admissible obtuse-angled triangles may proceed in a way similar to the one we used for acute-angled, delivering corresponding formulas for the area and the perimeter:

$$(ABC) = \frac{(h^2-(2x^2+r^2))\sqrt{4r^2-x^2}}{x}, \qquad 2\tau = 2\left(\sqrt{4r^2-x^2}+\sqrt{\frac{(2r-x)(h^2-(x-r)^2)}{x}}\right).$$

As suggested by figure 7, in this case the side-lines of the admissible triangles are tangent to a hyperbola with focals $\{H,O\}$. The critical, degenerated to segments, triangles are tangent to the hyperbola at its intersections $\{P',Q'\}$ with the circumcircle $\kappa'(O,2r)$ of the triangles. Again, by the symmetry of the figure, it is seen that all possible shapes of admissible triangles are obtained when B' varies inside the arc B_1B_2 of the circle κ, where B_1 is the intersection of κ and the segment NO and B_2 is the projection of O on QQ'. These correspond, for the variable $x = |OB'|$, to the endpoints of the interval

$$h - r \leq x \leq \sqrt{2(h^2-r^2)},$$

at which the area/perimeter obtains its maximum and minimum value. We formulate this as a corollary, showing that the upper bounds are the same for the two cases whereas the lower bounds are different.

Corollary 2. *The area (ABC) resp. perimeter 2τ of the triangle ABC with given Euler circle $\kappa(N,r)$ and distance from the orthocenter $|NH| = h > r$ is bounded by*

$$0 \leq (ABC) \leq (3r-h)\sqrt{4r^2-(r-h)^2}, \tag{5}$$
$$2\sqrt{18r^2-2h^2} \leq 2\tau \leq 2(\sqrt{r+h}+2\sqrt{r})\sqrt{3r-h}. \tag{6}$$

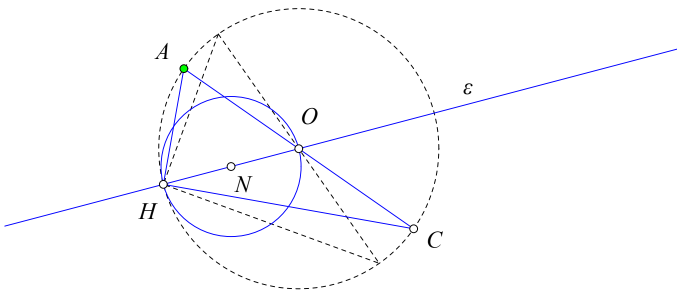

Figure 8: Admissible right-angled triangles

The case of right-angled admissible triangles, corresponding to the condition $h = r$ is indeed trivial. The admissible triangles are all right angled triangles inscribed in the same circle and having fixed the vertex H of the right angle. The maximal one is the isosceles and the minimal is degenerated to the diameter through H (See Figure 8).

4 A variational aspect of central conics

The discussion so far suggests an alternative aspect to the classical view of *"central"* conics. They are the envelopes of side-lines of an *"admissible family of triangles"*, which share their Euler circle $\kappa(N, r)$ and circumcircle $\kappa'(O, 2r)$. The first is the *"auxiliary circle"* of the conic, point N being the center of the conic. The second is the *"circular directrix"* of the

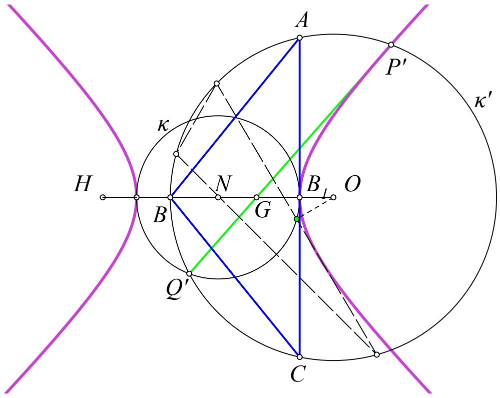

Figure 9: Central conics as envelopes of triangles sharing Euler- and circum-circle

conic, point O being one focus, the other focus coinciding with the common orthocenter H of all these triangles. Figure 9 shows the particular case of *"rectangular hyperbolas"*, characterized by the relation $h = |NO| = \sqrt{2}r$. By the discussion below and especially from equation (11) follows that all these triangles have a constant sum of squares of side-lengths $a^2 + b^2 + c^2$, which in the case of rectangular hyperbolas of figure 9 is easily seen to satisfy the relation

$$r^2 = \frac{1}{28}(a^2 + b^2 + c^2). \tag{7}$$

More generally, every central conic will determine such a maximal in area isosceles triangle ABC, whose area will bound the areas of all other triangles sharing the same Euler-, circum-circles $\{\kappa, \kappa'\}$. It will determine also an analogous to equation (7) for the sum of squares of side-lengths. Conversely, every isosceles triangle will define a conic and corresponding circumscribed to it triangles sharing with the isosceles its Euler-and its circum-circle. All these triangles will be obtuse if the apex-angle of the isosceles is obtuse and correspondingly acute-angled if the apex-angle is acute.

Theorem 4. *The non-rectangular triangles, having given radius r of the Euler circle and given sum of squares of side lengths $a^2 + b^2 + c^2 = k^2$, are congruent to the circumscribed triangles of a unique, up to congruence, central conic determined by $\{r, k\}$.*

Proof. By the preceding remarks, this reduces to showing, that there is, up to congruence, an isosceles inscribed in a circle of radius $R = 2r$ with given sum of squares of side-lengths $2a^2 + b^2 = k^2$. In fact, if we fix a circle of radius R and study the isosceli inscribed there, we see that the basis b and the preceding constant k, considered as functions of the lateral side a of the isosceles triangle, are given by the functions

$$b(a) = \frac{a}{R}\sqrt{4R^2 - a^2} \quad \text{and} \quad k(a) = \frac{a}{R}\sqrt{6R^2 - a^2}.$$

Figure 10 shows the graph of $k(a)$ and its relation to the varying isosceli inscribed in the circle of radius R. The maximal value of k is attained when the isosceles becomes an *equilateral*. The variation of k can be split into two parts.

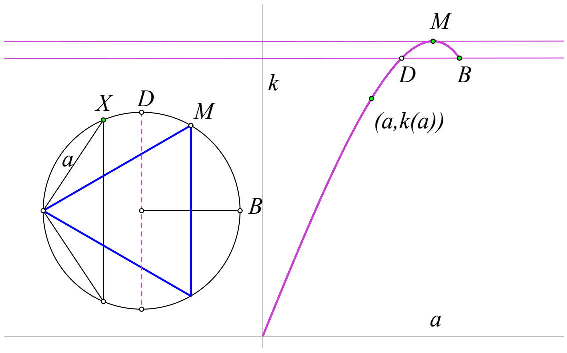

Figure 10: The constant k as a function of the lateral side a

The first part is connected with the *obtuse angled* triangles, for which the basis is on the left of the diameter at D. These isosceli are in one-to-one correspondence to the values of k. They correspond to families of obtuse-angled admissible triangles, which contain one only isosceles. Hence for obtuse-angled triangles, the value of k uniquely determines the corresponding isosceles contained in the family and through it the hyperbola enveloping the sides of all the triangles of the family.

The second part is connected with the *acute-angled* triangles, for which the basis is on the right of the diameter at D. These isosceli correspond two-to-one value of k, except the point M, which defines an equilateral. In the non exceptional case the isosceli correspond to families of acute-angled admissible triangles, which contain two such members. The maximal and the minimal one, studied in section 2. In this case, the value of k determines again a unique ellipse to which these triangles and all triangles of the family are circumscribed. Notice that the maximal value of k, obtained at M, is $3R$ and corresponds to the family of congruent equilaterals inscribed in the given circle, for which the ellipse coincides with their common Euler circle coinciding also with their incircle. □

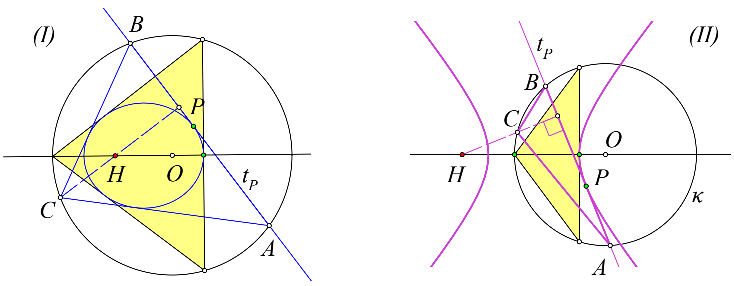

Figure 11: Constructing members of an admissible family of triangles

Figure 11 illustrates the construction of members of an admissible family created by the isoceles contained in it. The isoceles defines the conic and for points P, whose tangent t_P intersects the circumircle of the isosceles, the intersections $\{A, B\}$ define two vertices of the member-triangle. The third vertex C is one of the intersections of the circumcircle with the perpendicular from the orthocenter H of the isosceles to the line AB. In the case of the ellipse and acute-angled triangles, all tangents t_P intersect the circumcircle, whereas in the case of the hyperbola we have the restrictions discussed in section 2.

The two figures may be considered as a generalization of the familiar one of the equilateral triangle, rotating in its cicumcircle. The role of the incircle of the equilateral overtakes in the general case the ellipse for acute and the hyperbola for obtuse triangles.

5 Poristic triangles

The system $\mathcal{S}$ of *"admissible"*, in our context, triangles is traditionally called *"poristic"* in a wider sense. In fact, families of triangles sharing the same Euler circle and same circumcircle have been already considered by Weaver [17] and constitute a variation of the original idea of *"poristic triangles"*. Under this term discussed Gallatly initially in his book [4, p.23] triangles sharing the same *incircle* and same *circumcircle*. There are since then several articles discussing various aspects of such families of triangles. Among others, of some importance for our subject are the articles by Odehnal [9], Oxman [10] and Radic [14]. In this section we establish a connection with the original idea of poristic triangles formulated as a theorem.

Theorem 5. *Under the definitions and conventions adopted so far, the system $\mathcal{S}$ of acute-angled "admissible triangles", which share the same Euler circle κ and the same circumcircle κ', are the "intouch" triangles of a system $\mathcal{S}'$ of poristic triangles, coinciding with the tangential triangles of the former.*

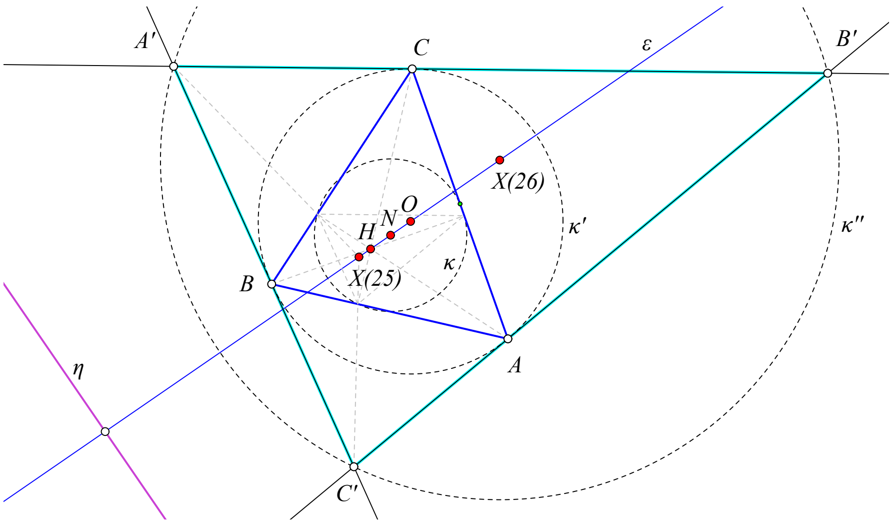

Figure 12: Relation of admissible ABC to *"poristic"* $A'B'C'$

Proof. The proof results by considering the tangential triangle $A'B'C'$ (See Figure 12) of ABC, which becomes then the *"intouch"* triangle of the former. Thus, it suffices to show that the circumcircle κ'' of $A'B'C'$ is the same for all admissible triangles. For this we have to show that the center of κ'' is a fixed point of the fixed Euler line ε of ABC and its radius is also constant. The first claim is a simple fact resulting from general well known properties of triangle centers, to be found in Kimberling's encyclopedia [7], [6]. In this list the center of the tangential triangle of ABC is denoted by $X(26)$ and seen to be a point of the Euler line ε. Expressed in barycentric coordinates or *"barycentrics"* ([12]) it has the representation

$$X(26) = \left(a^2(b^2\cos(2\widehat{B}) + c^2\cos(2\widehat{C}) - a^2\cos(2\widehat{A})) \ : \ \ldots \ : \ldots\right) \tag{8}$$

$$= (E+4F)(\ S^2\ ,\ S^2\ ,\ S^2\ ) - (3E+4F)(\ S_BS_C\ ,\ S_CS_A\ ,\ S_AS_B\ ) \tag{9}$$

$$\text{where} \quad E = 4R^2, \qquad F = S_\omega - 4R^2. \tag{10}$$

Here the first row expresses $X(26)$ in terms of the side-lengths $\{a, b, c\}$ and its angles $\{\widehat{A}, \widehat{B}, \widehat{C}\}$. The dots must be replaced with the same expression with cyclically permuted letter labels. The second row represents $X(26)$ as a linear combination of the barycentrics of the centroid $G(S^2 : S^2 : S^2)$ and the orthocenter $H(S_B S_C : S_C S_A : S_A S_B)$, the coefficients in the combination (9) being known as *"Shinagawa coefficients"* ([7]) of points on the Euler line. Here $\{S_A, S_B, S_C\}$ are the *"Conway triangle symbols"* of ABC, expressed by

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2) , \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2) , \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2) ,$$

and reviewed in [12]. The symbol S represents twice the area of ABC and

$$S_\omega = S\cot(\omega) = \frac{1}{2}(a^2 + b^2 + c^2),$$

with ω representing the *"Brocard angle"* of the triangle ABC ([3, p.274], [18, p.84], [12]). The symbols $\{E, F\}$ appearing in the aforementioned linear combination are in our case constant i.e. the same for all admissible triangles ABC. This because the circumradius $R = 2r$ is the constant radius of the circumcircle of ABC and S_ω enters the expression ([3, p.102]) of

$$OH^2 = 9R^2 - 2S_\omega, \tag{11}$$

showing its constancy, since $\{O, H\}$ and R are the same for all admissible triangles ABC.

As explained in [8], writing a triangle center as a linear combination $X = mG + nH$ with Shinagawa coefficients $\{m, n\}$, implies that point X is the image $X = h_{G,t}(H)$ of the homothety $h_{G,t}$ with center G and homothety-ratio

$$t = \frac{n}{3m + n}. \tag{12}$$

This, in the case of the linear combination (9), implies that the homothety ratio is constant:

$$t = -\frac{3E + 4F}{3(E + 4F) - (3E + 4F)} = -\frac{3E + 4F}{8F} .$$

Hence the location of $X(26)$ is the same for all triangles ABC of the system $\mathcal{S}$.

Analogous reasoning shows the constancy of the *"homothety center"* $X(25)$ of the tangential triangle $A'B'C'$ to the orthic triangle $A''B''C''$ of ABC :

$$\begin{aligned} X(25) &= \left(a^2 S_B S_C \ : \ \ldots \ : \ldots\right) & (13)\\ &= F(\ S^2\ ,\ S^2\ ,\ S^2\ ) - (E + F)(\ S_B S_C\ ,\ S_C S_A\ ,\ S_A S_B\ ) . & (14) \end{aligned}$$

The constancy of the radius R'' of κ'' follows from the known ([3, p.98]) homothety ratio r/R'' of the orthic triangle $A''B''C''$ to the tangential $A'B'C'$, which for the case of acute-angled ABC is given by ([5, p.259], [15]):

$$\frac{r}{R''} = 2\cos(\widehat{A})\cos(\widehat{B})\cos(\widehat{C}) = \frac{S_\omega - 4R^2}{2R^2} = \frac{1}{4R^2}(R^2 - OH^2). \tag{15}$$

□

Remark 2. It must be stressed the fact, that the theorem deals only with the case of *acute-angled* admissible triangles ABC. Indeed, all intouch triangles are acute, since their angles are of measure $\{\alpha = \frac{\pi - \alpha'}{2}\}$, where $\{\alpha'\}$ are the angle measures of the corresponding triangle $A'B'C'$, of which ABC is the intouch one (See Figure 13-I). In contrary, all the *"extouch"* triangles are obtuse-angled (See Figure 13-II). Thus, if we have a family of admissible obtuse-angled triangles $\{ABC\}$ and want a corresponding relation for this

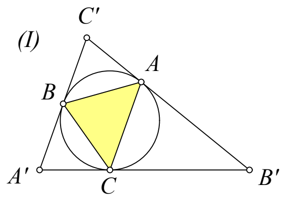

Figure 13: *"Intouch"* triangles: acute-angled, *"Extouch"* : obtuse-angled

case, we have to consider a variation of the original idea of poristic, concerning triangles with common circumcircle and one *"excircle"* of them. In the sequel we stick to the acute-angled case, leaving the variation of the *"poristic"* notion to pairs of (circumcircle , excircle) and corresponding relations to admissible families to be the subject of a future discussion.

Remark 3. Regarding the three circles $\{\kappa, \kappa', \kappa''\}$ of figure 12, it is known ([5, p.200]) that they belong to the same *"coaxal system"* ([3, p. 201]). Their common radical axis η is the *"orthic axis"* of ABC, i.e. the trilinear polar of the orthocenter H w.r. to ABC, which also coincides with the polar of $X(25)$ w.r. to the circumcircle κ' of ABC.

Remark 4. The system $\mathcal{S}$ of acute-angled admissible triangles being identical with the set of intouch triangles of a poristic system $\mathcal{S}'$ and the Euler line of $\mathcal{S}$ coinciding with the *"incenter-circumcenter-line"* of $\mathcal{S}'$, we can use $\mathcal{S}$ for the detection of fixed triangle-centers for all triangles of $\mathcal{S}'$. Especially for the triangle-centers that lie on the Euler line ε of $\mathcal{S}$ and for which we know their relation w.r. to ABC and $A'B'C'$. For example $X(25)$ w.r. to ABC is $X(57)$ of $A'B'C'$ when ABC is acute-angled, hence a fixed point of the poristic system $\mathcal{S}'$.

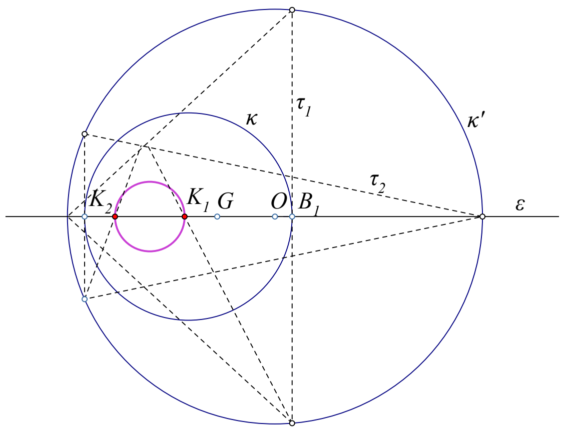

Figure 14: Geometric locus of symmedian points

Such correspondences between triangle centers w.r. to ABC and $A'B'C'$, could be used also for the detection of orbits of variable triangle centers of either system $\{\mathcal{S}, \mathcal{S}'\}$. For example, the *"symmedian point"* $K = X(6)$ of an acute-angled triangle ABC is the *"Gergonne point"* $X(7)$ of the corresponding tangential $A'B'C'$, which is known ([9], [2], [4, p.23]) to describe a circle coaxal with $\{\kappa', \kappa''\}$, as $A'B'C'$ varies in $\mathcal{S}'$. Hence the symmedian point K describes this same circle as ABC varies in $\mathcal{S}$. Figure 14 gives a visual hint for the location of the circle described by K for acute angled triangles. It has diameter on ε with ends the symmedian points $\{K_1, K_2\}$ of the corresponding maximal/minimal admissible triangles $\{\tau_1, \tau_2\}$.

Remark 5. The determination of maximal/minimal area/perimeter member-triangles in the poristic system $\mathcal{S}'$ has been handled in [14] with results similar to those of section 2. The maximal/minimal in area/perimeter triangles, also in this case are the isosceli members of $\mathcal{S}'$, having line ε for axis of symmetry.

6 Homography between two circles

Now, having a concept of the relation between admissible and poristic families of triangles, we turn to the study of a homography between two circles, having in mind the incircle and the circumcircle of an acute-angled triangle. Indeed, we'll adapt to our case the following more general homography, defined by means of a pair of circles $\{\kappa(r), \kappa'(r')\}$ and a line ζ intersecting each circle in two different points as in the figure 15. In this

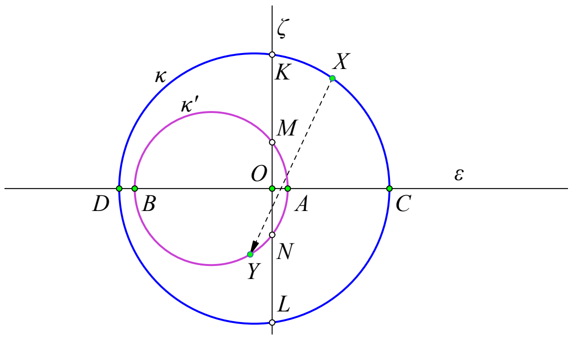

Figure 15: Homography mapping κ to κ'

ε is the line of centers and $\{(A, B), (C, D)\}$ are pairs of diametral points of $\{\kappa', \kappa\}$. By the *"fundamental theorem"* of projective geometry ([16, p.96]), there is a unique projective transformation (homography) f mapping:

$$K \overset{f}{\mapsto} N\,, \quad L \overset{f}{\mapsto} M\,, \quad C \overset{f}{\mapsto} B\,, \quad D \overset{f}{\mapsto} A\,.$$

Theorem 6. *With the notation and conventions adopted above, the homography f maps the circle κ onto the circle κ'.*

Proof. We use the cartesian coordinate system with axes $\{\varepsilon, \zeta\}$, representing the points by their associated homogeneous coordinates

$$K(0, k, 1)\,,\ M(0, m, 1)\,,\ N(0, -m, 1)\,,\ L(0, -k, 1)\,,\ C(c, 0, 1)\,,\ D(d, 0, 1)\,,\ B(b, 0, 1)\,,\ A(a, 0, 1)\,,$$

which satisfy the conditions

$$a \cdot b = -m^2\,, \quad c \cdot d = -k^2\,. \tag{16}$$

The homography is represented in the homogeneous system by a matrix $\mathcal{A}$, defined uniquely up to a non-zero multiplicative constand by the formal conditions

$$\mathcal{A} \cdot (KLC) = (NMB) \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix} \quad \text{and} \quad \mathcal{A} \cdot D = s \cdot A. \tag{17}$$

Here $\{(KLC), (NMB)\}$ are the matrices with column-vectors the corresponding to the labels coordinates and $\{u, v, w, s\}$ are factors to be determined. Inverting (KLC) we obtain, up to a multiplicative factor, the matrix

$$\mathcal{A} = (NMB)\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}(KLC)^{-1} = \begin{pmatrix} 2bkw & 0 & 0 \\ km(u-v) & -cm(u+v) & ckm(v-u) \\ k(2w-u-v) & c(u-v) & ck(u+v) \end{pmatrix}.$$

Using the last of equations (17) we see that the factors involved are, up to a common multiplicative constant,

$$u = v = \frac{b-a}{c-d} \quad \text{and} \quad w = \frac{a}{d},$$

leading finally to the, up to multiplicative factor expression of the matrix

$$\mathcal{A} = \begin{pmatrix} ab(c-d) & 0 & 0 \\ 0 & km(b-a) & 0 \\ ac-bd & 0 & k^2(a-b) \end{pmatrix}. \tag{18}$$

The proof results by showing that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \kappa \quad \Rightarrow \quad \mathcal{A}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in \kappa'. \tag{19}$$

In fact, taking into account equations (16), we see that the circles are represented by the equations

$$\kappa \;:\; x^2+y^2-(c+d)xz-k^2z^2=0, \qquad \kappa' \;:\; x'^2+y'^2-(a+b)x'z'-m^2z'^2=0.$$

Replacing in the second equation the variables $\{x', y', z'\}$ with their expressions given by equations (19) and doing some calculation, we find indeed that

$$x'^2+y'^2-(a+b)x'z'-m^2z'^2 = [(b-a)^2m^2k^2]\cdot[x^2+y^2-(d+c)xz-k^2z^2] = 0,$$

which finishes the proof. □

In the following remarks the equality between tripples of homogeneous coordinates, occasionally denoted by " $\cong$ ", is considered up to non-zero multiplicative factors, since (x, y, z) and (tx, ty, tz) for $t \neq 0$ represent the same point in the projective plane. By its definition, the homography f leaves invariant the lines $\{\varepsilon, \zeta\}$ and fixes their intersection point O. More generally, from its expression through the matrix $\mathcal{A}$, we can easily detect its behavior on lines parallel to the axes. In fact, for points $Y(0, y, z)$ of the y−axis, their image is $f(Y) = (0, my, -kz)$. Hence the point at infinity $(0, 1, 0)$ of this line remains fixed under f. This implies that a line ζ' parallel to ζ will map under f to a line ζ'' also parallel to ζ.

The behavior of f along the x−axis is also easily seen to be described by the correspondence

$$f(X(x,0,z)) = X' = (\, ab(c-d)x \,,\, 0 \,,\, (ac-bd)x - k^2(b-a)z \,),$$

which turning to the cartesian inhomogeneous coordinates $t = x/z$ leads to the representation by the *"fractional transformation"*

$$t' = g(t) = \frac{ab(c-d)t}{(ac-db)t + k^2(a-b)}. \tag{20}$$

From here we see that, besides the origin O corresponding to $t = 0$, the second fixed point P on ε corresponds to

$$t_0 \;=\; g(t_0) \qquad \Leftrightarrow \qquad t_0 \;=\; \frac{ab(c-d)+cd(a-b)}{ac-bd} \;,$$

defining the line η at P, orthogonal to ε. This is the third invariant line of the transformation f (See Figure 16) and $\{O, P\}$ are two fixed points of f, which together with

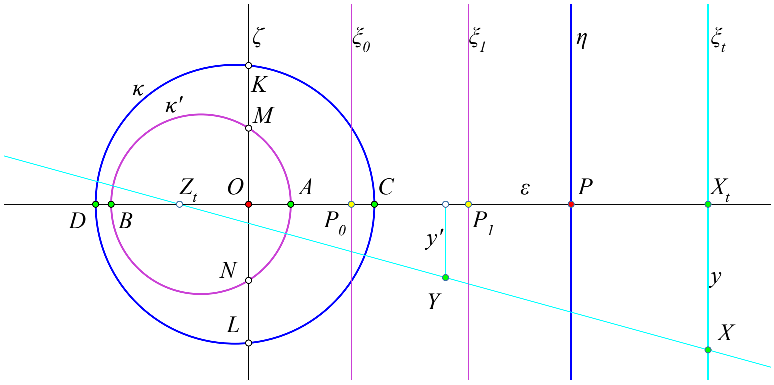

Figure 16: Some characteristic lines of the homography f

the point at infinity of ζ determine the three fixed points of f. Two other characteristic lines of f are the parallels $\{\xi_0, \xi_1\}$ to ζ defined by the points $\{P_0, P_1\}$ on the x−axis corresponding to the values of t :

$$t_0 \;=\; \frac{ab(c-d)}{ac-bd} \quad \text{and} \quad t_1 \;=\; \frac{cd(a-b)}{ac-bd} \;.$$

Line ξ_0 is the image under f of the line at infinity and line ξ_1 is the line send by f to infinity. It is readily seen that

$$|OP_0| \;=\; |P_1P| \;.$$

Point P_0 is the image under f of the point at infinity of ε. This implies that the parallels to ε, meeting this line at its point at infinity, have their images under f pass through P_0. More general, the images via f or parallels to a given direction pass through a point on line ξ_0.

The behavior of f along an arbitrary parallel ξ_t to ζ is characterized by the existence of a corresponding point $Z_t \in \varepsilon$, such that for all $X \in \xi_t$ the line joining $\{X, f(X)\}$ passes through Z_t. In fact, if ξ_t is characterized by the constant $t = x/z$, then the generic point on ξ_t is represented by $X(t, y, 1)$ with variable y and $f(X) \;=\; \mathcal{A}X$. By the preceding remarks, we know that $f(\xi_t)$ is a line parallel to ξ_t. Its points are described by

$$\begin{aligned} Y \;=\; \mathcal{A}X \quad &= (\,(ab(c-d)t\,,\; km(b-a)y\,,\; (ac-bd)t + k^2(a-b)\,) \\ \cong\; Y(x', y', 1) \;&=\; \left(\, \frac{ab(c-d)t}{(ac-bd)t + k^2(a-b)} \;,\; \frac{km(b-a)y}{(ac-bd)t + k^2(a-b)} \;,\; 1\,\right). \end{aligned}$$

In this the first coordinate x', recognized in equation (20), determines the projection of Y on ε and the second coordinate produces the depending on t only ratio

$$\frac{y'}{y} \;=\; \frac{km(b-a)}{(ac-bd)t + k^2(a-b)} \;,$$

proving the constancy of Z_t on ε.

7 On the Euler condition

The *"Euler condition"* known as *"Euler's theorem"* ([3, p.85])

$$OI^2 = R(R - 2\rho),$$

relates the distance of centers of two circles $\{\kappa(O, R),\ \kappa'(I, \rho)\}$ to their radii. It is the necessary and sufficient condition for the existence of triangles having the small circle (κ') as incircle and the big one (κ) as circumcircle. To be short, I call two such circles *"Euler conditioned"*. Figure 17 shows two such circles and a *poristic triangle*. At this stage

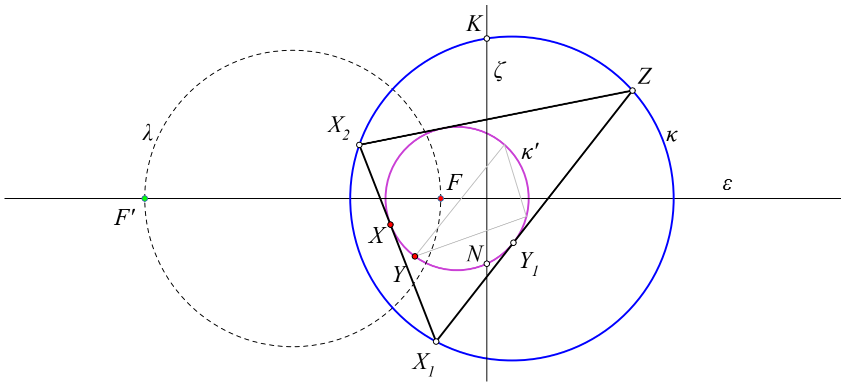

Figure 17: Triangle of a poristic system

we are interested in two correspondences noticed there. The first is the homography $Y = f(Z)$ defined by the two circles $\{\kappa, \kappa'\}$ and the line ζ in the way explained in the preceding section. The second correspondence $X = g(Z)$ associates to the vertex Z of the poristic triangle the contact point X of the opposite side with the incircle. In this section we deal mainly with two properties which I formulate as theorems.

Theorem 7. *The map $X = g(Z)$ is a homography of the circumcircle κ onto the incircle κ' of the Euler conditioned system $\{\kappa, \kappa'\}$.*

Theorem 8. *The homographies $\{f,\ g\}$ of an Euler conditioned system of circles $\{\kappa, \kappa'\}$ coincide, precisely when the line ζ passes through the inner limit point F of the coaxal system of circles defined by $\{\kappa, \kappa'\}$.*

Proof. (theorem 7) We use the cartesian coordinate system with the line ε as x−axis and the orthogonal to it at the center I of κ' as y−axis. Since the property is invariant by similarities, we may assume that the small circle κ' has radius $\rho = 1$ and the circle κ has its center at the point $(a, 0)$, so that the Euler condition becomes

$$a^2 = R^2 - 2R\,. \tag{21}$$

We proceed by constructing inversely $Z \in \kappa$ from $X \in \kappa'$ and thus showing that the inverse mapping of g is a homography. For this we follow the natural method to define Z by first locating the intersection points $\{X_1, X_2\}$ with κ of the tangent t_X at $X(x, y)$ and determining Z as intersection point of the two lines $\{X_1Y_1,\ X_2Y_2\}$, where $\{Y_1, Y_2\}$ are the reflections of X respectively on lines $\{IX_1,\ IX_2\}$. Since our circles are assumed Euler conditioned, we know that the such constructed point Z is on the circle κ and coincides with $g^{-1}(X)$. We use throughout vectors denoted by the same letters as the corresponding points they represent. The symbol $JX = (-y, x)$ denotes the $+\pi/2$ rotation about the origin I, satisfying $J^2 = -e$, where e the identity transformation and, being an isometry, preserving also the inner product $\langle JX, JY\rangle = \langle X, Y\rangle$. The points $\{X_1, X_2\}$ may be assumed in the form

$$X_1 \;=\; X \;+\; \lambda_1 JX, \qquad X_2 \;=\; X \;+\; \lambda_2 JX,$$

and using equation (21), the power of X w.r.t. to κ and the fact $x^2 + y^2 = 1$, we find that

$$(\lambda_1 \;+\; \lambda_2) \;=\; -2ay \quad \text{and} \quad \lambda_1\lambda_2 \;=\; 1 - 2ax - 2R\,.$$

A standard calculation of the reflected X on lines $\{IX_1, IX_2\}$ gives

$$Y_i \;=\; \mu_i X \;+\; \nu_i JX\,, \quad \text{with} \quad \mu_i \;=\; \frac{1-\lambda_i^2}{1+\lambda_i^2}\,,\; \nu_i \;=\; \frac{2\lambda_i}{1+\lambda_i^2} \quad \text{for} \quad i = 1, 2\,.$$

Point Z considered as intersection of the lines $Z = X_1Y_1 \cap X_2Y_2$, may be expressed in the form $Z = (1-t)X_1 + tY_1 = (1-t')X_2 + t'Y_2$. It is then determined by

$$Z \;=\; X_1 \;+\; t(Y_1 - X_1) \quad \text{with} \quad t \;=\; \frac{\langle\, X_2 - X_1\;,\; J(Y_2 - X_2)\,\rangle}{\langle\, Y_1 - X_1\;,\; J(Y_2 - X_2)\,\rangle} \quad \text{and}$$

$$X_2 - X_1 \;=\; (\lambda_2 - \lambda_1)JX\,, \qquad Y_i - X_i \;=\; (\mu_i - 1)X + (\nu_i - \lambda_i)JX\,, \quad \text{for} \quad i = 1, 2 \quad \Rightarrow$$

$$t \;=\; \frac{(\lambda_2 - \lambda_1)(\mu_2 - 1)}{(\nu_1 - \lambda_1)(\mu_2 - 1) - (\nu_2 - \lambda_2)(\mu_1 - 1)} \;=\; \frac{\lambda_2(1+\lambda_1^2)}{\lambda_1(1+\lambda_1\lambda_2)} \quad \Rightarrow$$

$$Z \;=\; \frac{1-\lambda_1\lambda_2}{1+\lambda_1\lambda_2}X \;+\; \frac{\lambda_1+\lambda_2}{1+\lambda_1\lambda_2}JX\,.$$

Expressing Z in homogeneous coordinates, and taking into account $x^2 + y^2 = 1$, we get

$$\begin{aligned} z_1 &= (1-\lambda_1\lambda_2)x - (\lambda_1+\lambda_2)y \;=\; 2(ax+R)x - (-2ay)y \;=\; 2a + 2Rx, \\ z_2 &= (1-\lambda_1\lambda_2)y + (\lambda_1+\lambda_2)x \;=\; 2(ax+R)y - 2ayx \;=\; 2Ry, \\ z_3 &= 1 + \lambda_1\lambda_2 \;=\; 2(1 - ax - R). \end{aligned}$$

This, homogenizing the right sides and dividing by 2, gives the representation of g^{-1} in homogeneous coordinates:

$$\begin{aligned} z_1 &= Rx \;+\; az\,, \\ z_2 &= Ry\,, \\ z_3 &= -ax + (1-R)z, \end{aligned}$$

proving it to be indeed a homography, as claimed. □

Proof. (theorem 8) I recall the classical result ([16, I,p.213]), according to which a homography $h : \kappa \to \kappa'$ between the points of two conics is completely and uniquely defined by prescribing the images $\{A', B', C' \in \kappa'\}$ of three points $\{A, B, C \in \kappa\}$. Now it is easily seen, on the ground of their proper definition, that all homographies $\{f\}$, depending on the location of the line ζ, coincide with g at the two diametral points $\{Z_1, Z_2\} = \kappa \cap \varepsilon$ of κ on line ε. The corresponding poristic triangles are then the two isosceli with the maximal/minimal area/perimeter. Thus, in order to prove the stated coincidence of the two homographies, it suffices to examine, when they are coincident at a single third point. This can be conveniently selected to be the point $N \in \kappa$ (See Figure 17), whose image per definition is $f^{-1}(N) = K$. Considering the system and the settings used in the preceding proof, g^{-1} can be described by the homography found there:

$$z_1 \;=\; \frac{Rx + a}{-ax + (1-R)}, \qquad z_2 \;=\; \frac{Ry}{-ax + (1-R)}.$$

Considering the coordinates $\{K(x,k), N(x,n)\}$, the coincidence condition of the images $\{g^{-1}(N) = f^{-1}(N) = K\}$ is equivalent with

$$x = \frac{Rx + a}{-ax + (1-R)}, \quad \text{and} \quad k = \frac{Rn}{-ax + (1-R)}. \tag{22}$$

The first of these implies that x must be equal to:

$$x = \frac{1}{2a}(1 - 2R + \sqrt{4R+1}), \tag{23}$$

which is the coordinate of the point F as claimed. Notice that this is one of the intersection points $\{F, F'\}$ of the minimal circle λ (See Figure 17) of the orthogonal pencil to the pencil of circles generated by $\{\kappa, \kappa'\}$. For the calculation of $\{F, F'\}$ in the context of circle pencils one can apply the formula ([13]) for their coordinates

$$x_F, x_{F'} = \frac{1}{d+c-a-b}\left(cd - ab \pm \sqrt{(c-a)(c-b)(d-a)(d-b)}\right),$$

the meaning of the constants being that of figure 15 discussed in section 6. □

Corollary 3. *Poristic systems of triangles $\mathcal{S}$ correspond bijectively to admissible systems $\mathcal{S}'$ of acute-angled triangles. Two corresponding systems $\{\mathcal{S}, \mathcal{S}'\}$ share the same Euler conditioned pair of circles $\{\kappa, \kappa'\}$. For the admissible system $\mathcal{S}'$, $\{\kappa, \kappa'\}$ are respectively the common circumcircle of the tangential triangles of the triangles $\tau' \in \mathcal{S}'$ and κ' is the common circumcircle of all $\tau' \in \mathcal{S}'$. For the poristic system $\{\kappa, \kappa'\}$ are the common circum- and in-circle of all triangles $\tau \in \mathcal{S}$. There is also a homography $g : \kappa \longrightarrow \kappa'$ mapping each triangle $\tau \in \mathcal{S}$ inscribed in κ to its intouch triangle $g(\tau) = \tau' \in \mathcal{S}'$.*

8 The Euler homography

From the analyzed properties of the homography g, defined in the preceding section, it seems reasonable to call it *"Euler homography"* of the *Euler conditioned* circles $\{\kappa, \kappa'\}$. In this section we explore further properties of g and relate them to properties of the two systems of triangles: the poristic $\mathcal{S}$ and the admissible $\mathcal{S}'$. In doing this we use the

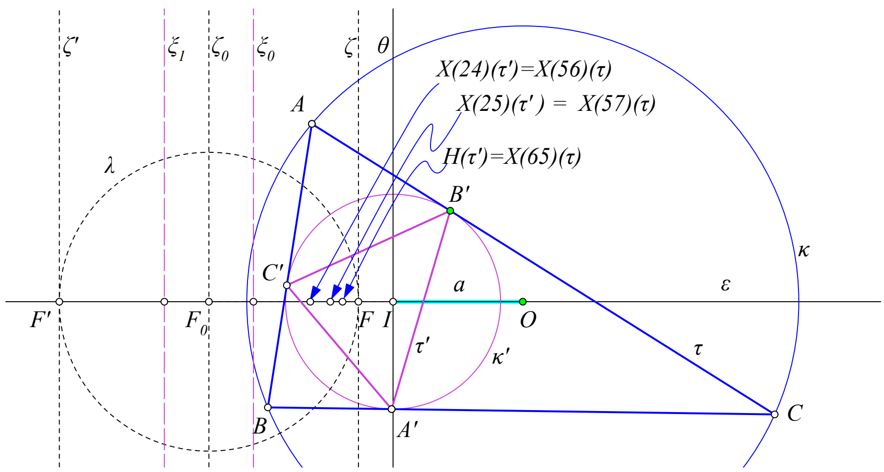

Figure 18: Properties of the Euler homography

notation and conventions of the preceding section about the coordinate system with axes

$\{\varepsilon, \theta\}$, the circles $\kappa(O, R)$ and $\kappa'(I, 1)$ (See Figure 18) and the distance $|IO| = a$, so that the Euler condition has the form of equation (21). We shall also apply the transformation g^{-1}, whose matrix representation w.r. to the aforementioned base has been found to be, up to non-zero factor:

$$g^{-1} \cong \begin{pmatrix} R & 0 & a \\ 0 & R & 0 \\ -a & 0 & 1-R \end{pmatrix} \quad \text{with inverse} \quad g \cong \begin{pmatrix} 1-R & 0 & -a \\ 0 & -1 & 0 \\ a & 0 & R \end{pmatrix}. \tag{24}$$

Since by theorem 8 the transformation g is a special case of the homographies studied in section 6, the properties discussed there apply also in the present case. From the first of the equations (22), which actually determine the fixed points of g^{-1}, we deduce that the points $\{F, F'\}$ with coordinates

$$x_F \ , \ x_{F'} \ = \ \frac{1}{2a}(1 - 2R \pm \sqrt{4R+1}) \ , \tag{25}$$

are fixed points of g. They are the diametral points on ε of the minimal circle $\lambda(F_0)$ among the orthogonal circles to $\{\kappa, \kappa'\}$. The third fixed point of g is the point at infinity F_∞ along the y–direction, so that $FF'F_\infty$ is the *"invariant triangle"* of g with vertices the fixed points and side-lines the invariant lines of g.

From the representations with matrices (24) we deduce further, that

$$x' \ = \ \frac{Rx + a}{ax + R - 1} \quad \text{and its inverse} \quad x \ = \ \frac{(1-R)x' - a}{ax + R} \ , \tag{26}$$

represent the restrictions of $\{g^{-1}, g\}$ on the line ε. From this we see that $x_0 \ = \ (1-R)/a$ and $x_1 = -R/a$ are the points send to infinity by g^{-1} resp. g, lying symmetrically w.r. to F_0. We deduce, that the lines $\{\xi_0, \xi_1\}$ orthogonal to ε at these points are respectively, the image via g of the line at infinity, and the line send to infinity by g. By the discussion in section 5 we know that the line ζ_0, orthogonal to ε at the center F_0 of the circle λ, is the *"orthic axis"* of the triangle $\tau' = A'B'C'$, i.e. the trilinear polar of the orthocenter $H(\tau')$ w.r. to triangle τ' coinciding also with the polar of $X(25)(\tau')$ w.r. to κ'. As we noticed already in section 5, $H(\tau')$ coincides also with the triangle center $X(65)(\tau)$.

Theorem 9. *The homography g maps the incenter I of $\tau = ABC$ to the orthocenter H of the intouch triangle $\tau' = A'B'C'$ and the orthocenter H to the Euler-circle center N of τ'.*

Proof. Since ε is invariant under g, we can use the second of the *fractional transformations* of equations (26), which applied to $I(0,0)$ gives for the coordinate of

$$H: \ x_H \ = \ -a/R \ = \ -\sqrt{R^2 - 2R}/R \ = \ -\sqrt{1 - \frac{2}{R}} \ .$$

But this is precisely the signed distance HI computed by formula (15) involving there the data $\{r, R, R''\}$, which in our configuration are correspondingly $\{1/2, \ 1, \ R\}$. The second claim follows by applying the same fractional transformation to the coordinate $-a/R$ of H giving the coordinate $-a/(2R)$ representing the middle of IH, which is indeed the Euler center N of τ'. □

Theorem 10. *The lines $\{\xi_1, \xi_0\}$ are respectively the polars w.r. to κ' of the points $\{H, X(24)\}$ relative to τ'.*

Proof. For this it suffices to show that the coordinates $\{x_1,\ x_0\}$ of the corresponding intersections $\{\xi_1 \cap \varepsilon,\ \xi_0 \cap \varepsilon\}$ are correspondingly inverse w.r. to κ' of the coordinates $\{x_h, x_{24}\}$ of $\{H,\ X(24)\}$. For H this is immediately seen from the previous computations, which imply indeed that $x_1 \cdot x_H = 1$. For $X(24)$ this can be seen by using the *Shinagawa coefficients* $\{2F,\ -E-2F\}$ of $X(24)$ ([7]) and doing a standard computation, as we did in section 5, now considering the circumcircle $\kappa'(I,1)$ and the circumcircle $\kappa(0,R)$ of the tangential triangle:

$$x_H = -\sqrt{1-\frac{2}{R}} \quad \Rightarrow \quad IH^2 = 1-\frac{2}{R} = 9-2S_\omega \quad \Rightarrow$$

$$S_\omega = \frac{1}{R}+4 \quad \Rightarrow \quad E = 4 \quad \text{and} \quad F = \frac{1}{R} \quad \Rightarrow$$

$$m = 2F = \frac{2}{R}, \quad n = -E-2F = -4-\frac{2}{R}, \quad \Rightarrow \quad t = \frac{n}{3m+n} = \frac{2R+1}{2R-2} \quad \Rightarrow$$

$$X(24) = h_{G,t}(H) = G + t(H-G) = (1-t)G + tH = \frac{1}{2R-2}(-3G + (2R+1)H)$$

$$\text{with} \quad x_G = -\frac{1}{3}\sqrt{1-\frac{2}{R}} \quad \Rightarrow \quad x_{24} = \frac{R}{1-R}\sqrt{1-\frac{2}{R}} = \frac{a}{1-R}.$$

which shows that $x_{24} \cdot x_0 = 1$. □

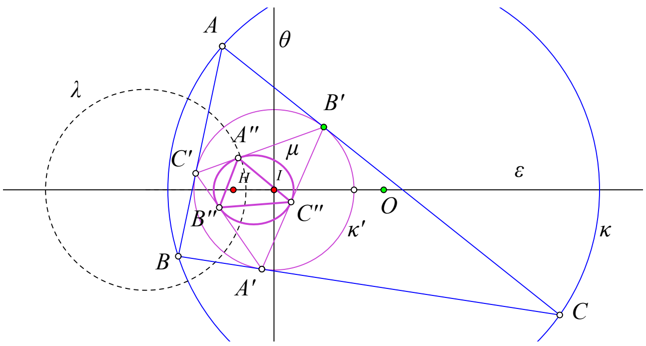

Figure 19: The Euler homography maps κ' to μ

Theorem 11. *The Euler homography g maps the circumcircle κ' of the triangle $A'B'C'$ to the ellipse μ with focals the circumcenter I and the orthocenter H of the triangle $A'B'C'$. The homography maps also the vertices $\{A', B', C'\}$ and the tangents there to corresponding vertices of its intouch triangle $A''B''C''$ and the corresponding tangents at these points (See Figure 19). The focal poins $\{I, H\}$ of μ are correspondingly images under g of the triangle centers $\{X(24), I\}$ of $A'B'C'$.*

Proof. The first part, concerning the correspondence of triangles $\{A'B'C', A''B''C''\}$ under g is a consequence of the properties of homographies and the corresponding correspondence of triangles $\{ABC, A'B'C'\}$. The property $g(I) = H$ was proved in theorem 9. The property $g(X(24)) = I$ is immediately seen by applying the second of the fractional transformations in equation (26) to the coordinate $x_{24} = \frac{a}{1-R}$ of $X(24)$. □

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