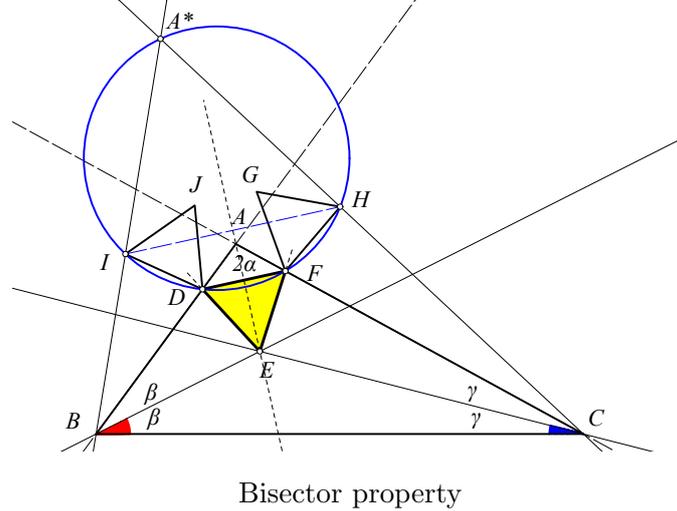


A short proof of Morley's theorem

Paris Pamfilos
 University of Crete, Greece
 pamfilos@uoc.gr

Among the dozens of proofs of Morley's Theorem, the proofs of J.M.Child [J.M23] and Naraniengar [HC67, p.47] are particularly neat. In this paper, we simplify them by stressing the symmetry of the equilateral and the symmetries created by two reflections.¹



For this, we start with a trivial property of the bisectors of a triangle ABC with angles $\{2\alpha, 2\beta, 2\gamma\}$.

Lemma. *From the incenter E of triangle ABC and on both sides of AE draw two lines inclined to it by 30° and intersecting the other sides, respectively at D, F . Then DEF is equilateral and DF is orthogonal to AE .*

Having that, reflect BE and DEF on AB to obtain BI and the equilateral DIJ . Do the same on the other side, i.e. reflect CE and DEF on CF to obtain CH and the equilateral FHG . By the symmetry w.r. to AE the quadrilateral $IDFH$ is an isosceles trapezium with three equal sides, hence cyclic, and its angle \widehat{IHF} is easily computed:

$$\begin{aligned}\widehat{IDF} &= \widehat{IDJ} + \widehat{JDF} = 60^\circ + 2(90^\circ - \alpha) = 60^\circ + 180^\circ - 2\alpha = 60^\circ + 2\beta + 2\gamma \Rightarrow \\ \widehat{IHF} &= 180^\circ - \widehat{IDF} = 120^\circ - 2\beta - 2\gamma.\end{aligned}$$

This, essentially, finishes the proof, since the arcs (IDF) and $(IDFH)$ are in ratio 2:3 and the angle at the intersection point A^* of lines BI and CH will be then of measure $\widehat{A^*} = 180^\circ - 3\beta - 3\gamma$. This shows that A^* is on the circumcircle κ of $IDFH$ and $\{A^*D, A^*F\}$ are the trisectors of $\widehat{A^*}$, implying obviously that, “for a triangle A^*BC with base angles $\{3\beta, 3\gamma\}$ the adjacent trisectors intersect at the vertices of an equilateral triangle”, as required by the theorem of Morley.

References

- [HC67] S Greitzer H Coxeter. *Geometry Revisited*. Mathematical Association of America, Washington D.C., 1967.
- [J.M23] J.M.Child. Proof of “Morley's theorem”. *Mathematical Gazette*, 11:171, 1923.

¹I would like to express my gratitude to the referee, for pointing out to me the proof by J.M.Child and making several useful remarks, helping me towards a better and more concise exposition.