A Characterization of the Focals of Hyperbolas

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Abstract

In this article we prove a property characterizing the focal points of hyperbolas.

1 Chords through a point

The property which we discuss here relates to the tangents of a hyperbola at the end points of a chord and their intersections with the asymptotes of the hyperbola. It is formulated by the following lemma.

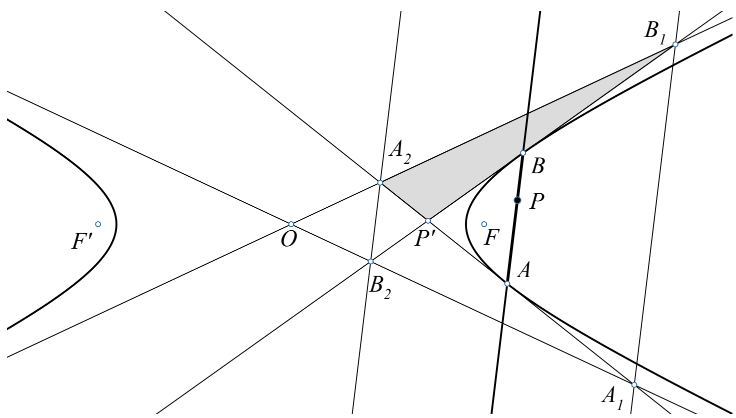

Figure 1: Asymptotic triangles and parallels

Lemma 1. *If the tangents to the hyperbola at the end points of a chord AB intersect the asymptotes respectively at points $\{A_1, A_2\}$ and points $\{B_1, B_2\}$. Then $\{A_1B_1, A_2B_2\}$ are parallels and AB is their middle-parallel.*

Proof. The proof of the lemma, in the case AB runs in the *inner* domain of the hyperbola (See Figure 1), derives from the equality of the areas of the triangles $\{A_1A_2B_1, A_1B_2B_1\}$, which have in common the area of the triangle $A_1P'B_1$, and are complemented by the equal areas of the triangles $\{P'A_2B_1, P'B_2A_1\}$ ([3, III.43, p. 112], [5, p.192]), point P' being the intersection of the tangents. The claim about the middle-parallel follows from the equally well known property ([3, II.3, p. 56], [4, fig. 10.18, p. 315], [5, p.191]), that $\{A, B\}$ are respectively the middles of $\{A_1A_2, B_1B_2\}$. The proof, when AB runs in the *outer* domain of the hyperbola is completely analogous[1] □

[1]At this point I would like to express my gratitude to the referee, who kindly suggested not only the references to the classical literature, but also a complete alternative proof to the main theorem. I hope to see this proof, as well as some other, possibly better or simpler proofs, from interested readers, published in this journal.

2 The property of focal points

Next theorem, characterizes the focal points $\{F, F'\}$ of the hyperbola by measuring the distance of the parallels $\{A_1B_1, A_2B_2\}$, as the chord AB turns about a fixed point P.

Theorem 1. *Under the notation and conventions made above, for chords passing through a fixed point P, the distance between the parallels $\{AB, A_1B_1\}$ is variable, depending on their direction, except when P is a focal point. In the case P is a focal point, this distance is independent of the direction and equal to the conjugate axis b of the hyperbola*

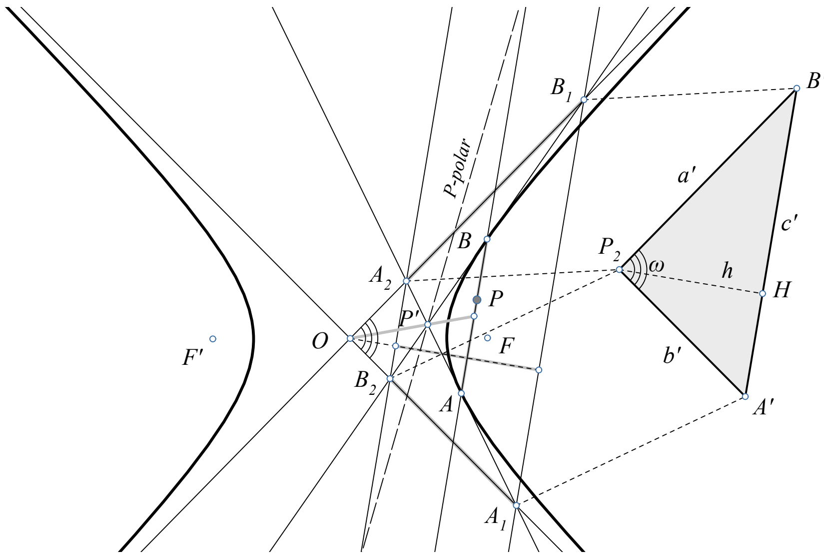

Figure 2: Triangle formed by the segments cut on the asymptotes

Proof. To prove this, we represent the hyperbola with its canonical coordinates in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

We consider also the quadratic equation, giving the product of the tangents from the point $P'(x_1, y_1)$. This can be seen to be ([2, p.251, I])

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2. \tag{1}$$

The intersection points $\{A_2, B_1\}$ and $\{B_2, A_1\}$ of these lines with the asymptotes are found by solving the systems consisting of the previous equation and the equation of each asymptote $x/a - y/b = 0$ and $x/a + y/b = 0$ (See Figure 2). These are found to be

$$A_2, B_1 = \frac{-ab \pm g}{ay_1 - bx_1}(a, b) \qquad \text{and} \qquad B_2, A_1 = \frac{ab \pm g}{ay_1 + bx_1}(a, -b), \tag{2}$$

where, $g = g(x_1, y_1) = \sqrt{a^2y_1^2 - b^2x_1^2 + a^2b^2}$. This implies that

$$|A_2B_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 - bx_1)^2} \qquad \text{and} \qquad |B_2A_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 + bx_1)^2}. \tag{3}$$

The required distance h of the parallels can be measured from the altitude of the triangle $P_2A'B'$, resulting by parallel translating at an arbitrary point P_2 the segments

$\{A_2B_1, B_2A_1\}$. Since the property under consideration is invariant by similarities, we can assume that $a^2+b^2=1$. Thus, using the well known formula, deriving from the area of a triangle, $h=\frac{b'c'\sin(\omega)}{a'}$, we find that

$$h^2 = \frac{b'^2c'^2\sin(\omega)^2}{a'^2} = \frac{2(a^2y_1^2 - b^2x_1^2 + a^2b^2)\sin(\omega)^2}{a^2y_1^2 + b^2x_1^2 + (a^2y_1^2 - b^2x_1^2)\cos(\omega)}, \tag{4}$$

where ω is the angle of the asymptotes. Taking into account that $\sin(\omega) = 2ab$, and $\cos(\omega) = a^2 - b^2$, we obtain the simplified expression

$$h^2 = 4a^2b^2\frac{a^2y_1^2 - b^2x_1^2 + a^2b^2}{a^4y_1^2 + b^4x_1^2}. \tag{5}$$

Letting the chord AB revolve about $P(x_0, y_0)$, the corresponding point $P'(x_1, y_1)$ moves on the polar line $\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$ of P ([1, p.192]), a particular point of which is $K_2(x_2, y_2) = (a^2/x_0, 0)$. A parametric form of the polar is consequently given by

$$x_1 = \frac{a^2}{x_0} + t\frac{y_0}{b^2}, \quad y_1 = t\frac{x_0}{a^2}.$$

Introducing this into equation-(5) and simplifying, we obtain

$$\begin{aligned} h^2 &= 4\frac{p(t)}{q(t)}, \quad \text{with} \\ p(t) &= t^2[-x_0^2(a^2y_0^2 - b^2x_0^2)] + t[-2a^4b^2x_0y_0] + [a^4b^4(x_0^2 - a^2)], \\ q(t) &= t^2[x_0^2(x_0^2 + y_0^2)] + t[2a^2b^2x_0y_0] + [a^4b^4]; \end{aligned}$$

The condition of constancy of h^2 is equivalent with the vanishing of coefficients of the quadratic equation $p(t) - kq(t)$, for a constant k, which implies the equations

$$\begin{aligned} x_0^2(y_0^2(a^2+k) - x_0^2(b^2-k)) &= 0 \\ (a^2+k)x_0y_0 &= 0, \\ (x_0^2 - a^2) - k &= 0. \end{aligned}$$

the two last equations, for $x_0y_0 \neq 0$ lead to contradiction. The condition $x_0 = 0$, leads also to the contradiction $h^2 = -4a^2$. Thus, if a point (x_0, y_0) has the stated property, it must satisfy $y_0 = 0, x_0 \neq 0$, implying $k = b^2 = (x_0^2 - a^2)$, hence $x_0^2 = 1$, which determines the position of a focal point $F(\pm 1, 0)$ and the value for $h^2 = 4b^2$, which proves the theorem. $\square$

References

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