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Elemente der Mathematik

Symmedian and isodynamic points relations

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1 Introduction

Let ABC be a triangle, let D be a point which is different from the points $\{A, B, C\}$ and let $\{A'', B'', C''\}$ be respectively the intersections of the lines AD with BC, BD with CA, and CD with AB (see Figure 1). Finally, let $\{A', B', C'\}$ be respectively the intersections of BC with $B''C''$, CA with $C''A''$ and AB with $A''B''$. Then, by Desargues' Theorem, $\{A', B', C'\}$ are collinear. Recall that Desargues' Theorem states that *"if the lines $\{AA'', BB'', CC''\}$ intersect at a point D then the points $\{A', B', C'\}$ are collinear and vice versa"*.[1] With respect to the triangle ABC, the line ε_D through the so-called *"homologous points"* $\{A', B', C'\}$ is called the *"trilinear polar"* or *"tripolar"* of the point D, and D is called the *"tripole"* of the line ε_D. An example of trilinear polar that we use below is represented by the *"Lemoine axis"* of the triangle. This is the trilinear polar of the *symmedian point* K of the triangle, whose traces $\{A'', B'', C''\}$ on the sides of ABC are defined by the *symmedians* of the triangle. The *"symmedian"* AA''

[1]At this point I would like to express my gratitude to the referee, whose suggestions greatly contributed to the improvement of the presentation.

Die Brocard-Achse eines Dreiecks ABC verbindet die beiden Schnittpunkte der drei Apollonius-Kreise des Dreiecks, die sogenannten isodynamischen Punkte, und verläuft gleichzeitig durch den Umkreismittelpunkt und den Lemoine-Punkt, der auch Symmedianenpunkt oder Grebe-Punkt genannt wird. Die Lemoine-Gerade des Dreiecks ABC ist dann die trilineare Polare des Lemoine-Punktes und steht senkrecht auf der Brocard-Achse. Der Autor der vorliegenden Arbeit geht der Frage nach, ob eine analoge Konfiguration existiert, wenn man den Lemoine-Punkt durch einen beliebigen andern Punkt der Ebene ersetzt und findet dabei einen geeigneten Ersatz für die Apollonius-Kreise.

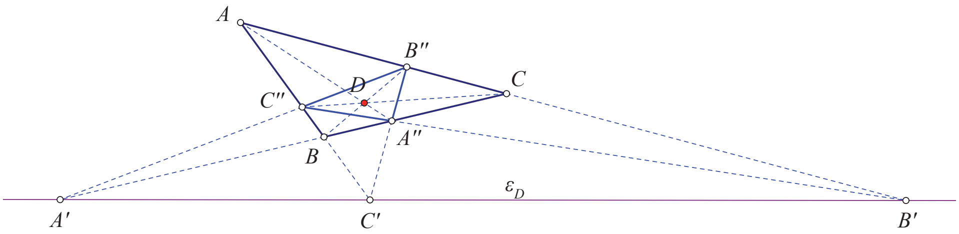

Figure 1 The trilinear polar ε_D of D w.r.t. ABC (Color figures online)

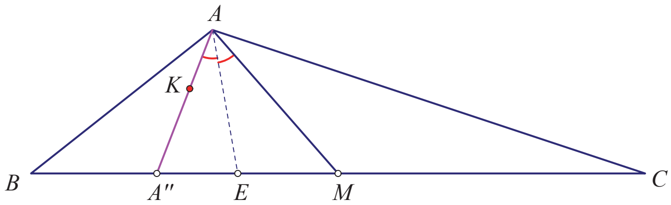

Figure 2 The symmedian AN from A (Color figures online)

from A is related to the *median* AM of the triangle from A (see Figure 2) and is created by reflecting AM in the bisector AE of $\widehat{A}$. It is proved that the three symmedians from $\{A, B, C\}$ concur at a point, the *"symmedian point"* K of the triangle, whose tripolar is the so-called *"Lemoine axis"* of the triangle (see Figure 3). The Lemoine axis is im-

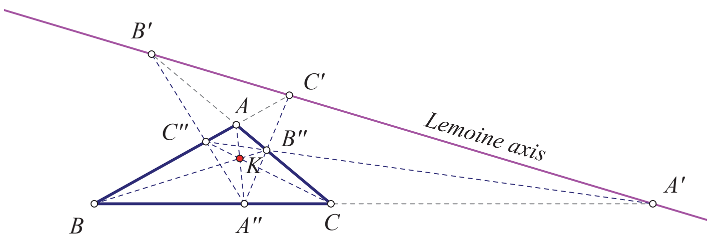

Figure 3 The *"Lemoine axis"* of ABC (Color figures online)

portant in our context, since in this article we study a generalization of the *"Apollonian circles"* $\{\kappa_A(A', |AA'|), \kappa_B(B', |BB'|), \kappa_C(C', |CC'|)\}$ of the triangle ABC. The Apollonian circle κ_A is defined as the geometric locus of points X such that the ratios of distances XB/XC is constant and equal to AB/AC. Analogously are defined the two other Apollonian circles. It is known that these three circles intersect pairwise at angles of measure $\pi/3$ at the two *"isodynamic points"* of the triangle $\{I_1, I_2\}$ (see Figure 4), which lie symmetrically w.r.t. the Lemoine axis and their line, called *"Brocard axis"* of the triangle, passes through the circumcenter O and the symmedian point K. Thus, the three circles belong to the same *"pencil of coaxal circles"* of *"intersecting type"*. By definition, a *pencil of coaxal circles* consists of a family of circles, for which there is a line ε, called

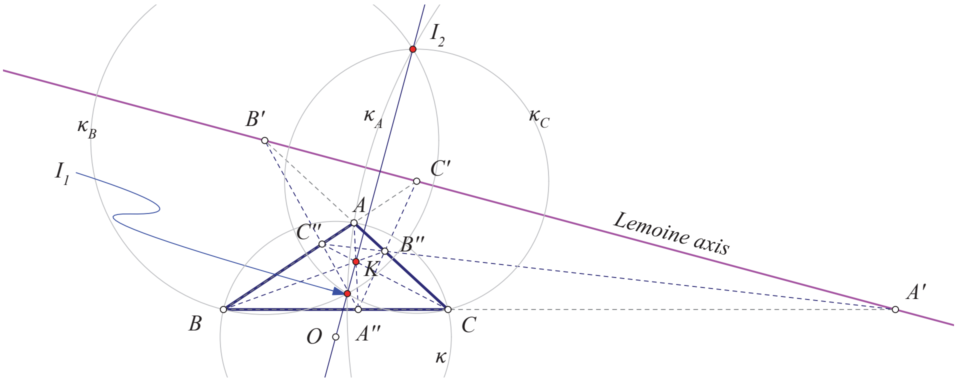

Figure 4 The isodynamic points $\{I_1, I_2\}$ (Color figures online)

the *"radical axis"* of the pencil, which is the radical axis of every pair of circles belonging to the family. In our case this is the *Brocard axis* OK. The centers of the circles of the pencil lie on a line orthogonal to the radical axis of the pencil, called *"line of centers"* of the pencil, which in our case is the *Lemoine axis*. Trilinear polars play an important

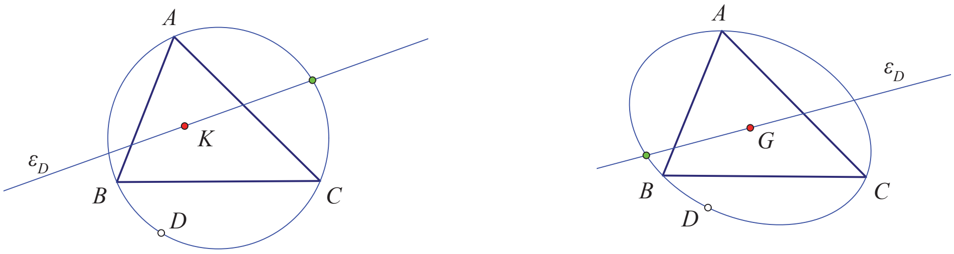

Figure 5 Circumcircle and Steiner ellipse (Color figures online)

role in the *"geometry of the triangle"* ([10], [21]) since they are closely related to the generation of conics circumscribing the triangle, i.e., conics passing through its vertices or conics inscribed in the triangle, i.e., tangent to its sides. Figure 5 shows two examples. The first is the circumcircle of the triangle. It is generated as the locus of tripoles D of all lines through the symmedian point K of the triangle. The second example is the *"Steiner ellipse"* of the triangle generated by the tripoles D of all lines through the centroid G of the triangle and having its center at G.

All the above ideas and properties are well known and, as far as elementary properties of the triangle are concerned, two standard references are [5] and [14]. For properties of conics circumscribing or inscribed in a triangle and their relations to tripolars a concise exposition can be found in [21] and [10].

In this article we will define and study a new configuration (see Figure 6), generalizing that of Apollonian circles of Figure 4, by replacing the symmedian point K with an arbitrary point D of the plane of the triangle.

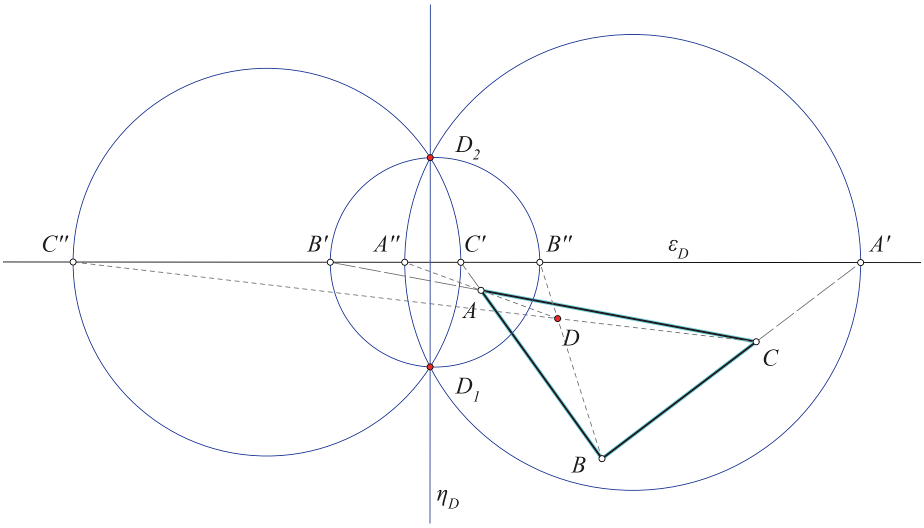

Figure 6 The trilinear isodynamics $\{D_1, D_2\}$ of D (Color figures online)

Definition 1. Given a triangle ABC and a point D on its plane, not contained in some side-line of the triangle, the *"trilinear pencil of D"* is defined to be a set of three circles, called *"trilinear isogonal circles of D"*, belonging to a pencil of intersecting type, intersecting pairwise at the same angle of measure $\pi/6$, and whose line of centers is the trilinear polar ε_D of of D. The basic points $\{D_1, D_2\}$ of the pencil are called *"trilinear isodynamics of D"* and their line is called the *"trilinear radical of D."*

In the following sections we show the existence of such a pencil depending from the point D and prove, that when D coincides with the symmedian point K of ABC, then the corresponding *trilinear isodynamics of D* coincide with the *isodynamic points* of the triangle, though the *trilinear isogonal circles of D* do not coincide with the Apollonian circles of the triangle. We prove also, that conversely, if the *trilinear isodynamics of D* coincide with the usual isodynamics of the triangle, then point D is the symmedian point of the triangle.

The technical side of all this relays on a known alternative to Ceva's theorem for the coincidence of three lines through the vertices of the triangle ABC, handled in Section 2. In Section 3 we apply this *alternative* to show the existence of the *trilinear isogonal circles* of D. In Section 4 we determine the geometric locus of all points $\{D : D \in \eta_D\}$, i.e., the locus of points D, that happens to lie on their trilinear radical ε_D. It is a curve of fifth degree, known as the *"Stother's quintic"*. This curve (see Figure 14) passes through the vertices, the middles of the sides, the centroid and the symmedian point K of the triangle. Finally in Section 5 we show the coincidence of the trilinear isodynamics of K with the traditional isodynamic points of the triangle of reference.

2 Coincidence criterion alternative to Ceva's one

This is a known exercise ([20, I, p. 51],[17, p. 319]) based on other exercises, appearing as applications of the *"Desargues' involution theorem"* for conics ([18, p. 311], [4, p. 381]). For convenience I discuss here all the fundamental ideas involved. The core of this subject is the *"complete quadrangle"*, defined by four points in general position and the *"quadrangular set"*, defined by a complete quadrangle on an arbitrary line.

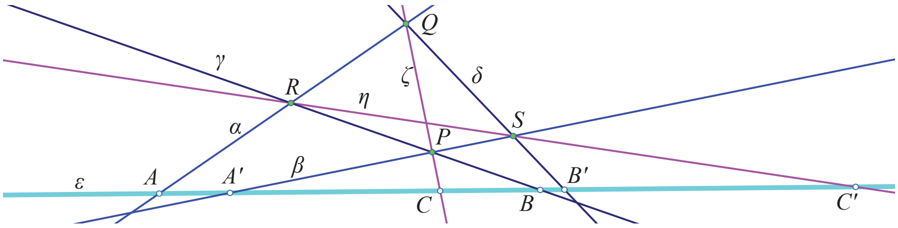

Figure 7 Three pairs in involution $\{(A, A'), (B, B'), (C, C')\}$ (Color figures online)

The *"complete quadrangle"* consists of four points $\{P, Q, R, S\}$ in general position, called *"vertices"* and the six lines they define by two:

$$\alpha = QR, \quad \beta = PS, \quad \gamma = PR, \quad \delta = QS, \quad \zeta = QP, \quad \eta = RS,$$

called *"sides"* of the complete quadrangle. Two sides (as, f.e., $\{\alpha, \beta\}$ in Figure 7), not containing the same vertex, are called *"opposite"*.

A complete quadrangle, with vertices $\{P, Q, R, S\}$, and an arbitrary line ε, that does not pass through these points, define a *"quadrangular set"* ([6, p. 20]). This consists of three pairs of points

$$\{(A, A'), \ (B, B'), \ (C, C')\},$$

each pair being defined through the intersections of ε with a *pair of opposite sides* of the complete quadrangle (see Figure 7).

Quadrangular sets are closely related to *"involutions of points on a line"*. An *involution* is a transformation of the projective line ε onto itself, described in homogeneous coordinates $\{x, y\}$ and their ratio $t = x/y$ by a rational function

$$t' = f(t) = \frac{a \cdot t + b}{c \cdot t - a} \quad \text{with} \quad a^2 + bc \neq 0. \tag{1}$$

The point of the line send by the involution to infinity and corresponding to $t = a/c$ is called the *"center"* of the involution. Two points $Y = f(X)$ related by an involution are called *"conjugate"* by f.

We recall that the *"projective coordinates"* on a projective line are defined by fixing three pairwise different points $\{A, B, C\}$ of it, called *"basic"*, to which we associate respectively the coordinates $\{(1, 0), (0, 1), (1, 1)\}$. Every other point P of the line is then described by a pair of numbers $(x, y) \neq (0, 0)$, defined up to non-zero multiplicative constant. Thus, (x, y) and (kx, ky) , $k \neq 0$, define the same point ([2, p. 45]), which is represented formally as a linear combination $xA + yB$.

Three fundamental properties of involutions are ([6, p. 45], [8, p. 262]), that (i) they are completely determined by prescribing two points and their images, (ii) they preserve the cross-ratio of four points and, as their name suggests, (iii) they are *involutive* $\{f^2 = id \Leftrightarrow f^{-1} = f.\}$ By property (i), two pairs of points $\{(A, A'), (B, B')\}$, define a unique involution f mapping $\{f(A) = A', f(B) = B'\}$. In general, $f(C) \neq C'$. If it happens that $f(C) = C'$, then we say that the three pairs $\{(A, A'), (B, B'), (C, C')\}$ of collinear points *"are in involution"*.

The well-known *"involution theorem of Desargues"* ([6, p. 46], [1, p. 145]) states, that *"The pairs* $\{(A, A'), (B, B'), (C, C')\}$ *of a quadrangular set are in involution. Conversely, three pairs in involution on a line* ε *are defined as a quadrangular set of an appropriate complete quadrangle."*

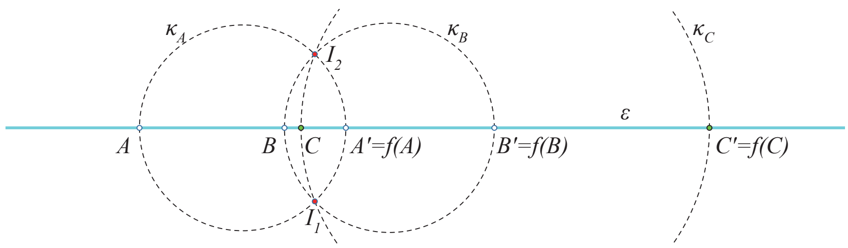

Figure 8 Involutions define pencils of circles (Color figures online)

Remark 1. We can describe geometrically an involution f on a line ε by associating to a pair of points $\{A, A' = f(A)\}$ the circle κ_A with diametral points $\{A, A'\}$ (see Figure 8). It turns out ([1, p. 140]), that these circles are the members of a coaxal pencil. If the pencil is of intersecting type, as in the figure, then the involution has two *imaginary* fixed points. If the pencil is of non-intersecting type, then the involution has two *real* fixed points.

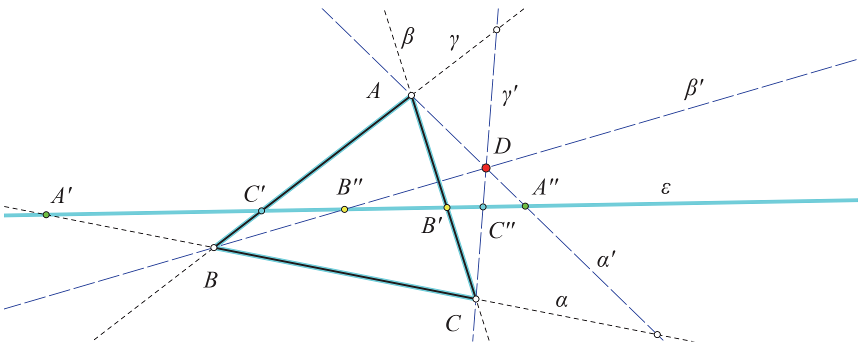

Figure 9 Alternative to Ceva's theorem (Color figures online)

The next lemma formulates the alternative Ceva criterion, whose proof uses a trivial sort of converse to Desargues' theorem. The lemma guarantees the concurrence of three lines

$\{\alpha', \beta', \gamma'\}$ at a point D (see Figure 9). The three lines are assumed to pass through corresponding vertices $\{A, B, C\}$ of the triangle ABC and also assumed to intersect an arbitrary but fixed line ε correspondingly at the points $\{A'', B'', C''\}$. In addition, the four points $\{A, B, C, D\}$ are assumed to be in *"general position"*, i.e., no three of them be collinear, and the line ε assumed not to pass through any of them.

Lemma 1 (Alternative to Ceva's theorem)**.** *Under the preceding assumptions, the three lines* $\{\alpha', \beta', \gamma'\}$ *concur at a point* D, *if and only if the pairs of points* $\{(A', A''), (B', B''), (C', C'')\}$ *are in involution.*

Proof. If the lines $\{\alpha', \beta', \gamma'\}$ concur at a point D, under the assumptions of the lemma, then by Desargues' involution theorem applied to the complete quadrangle $ABCD$, the three pairs of points are in involution. Conversely, assume that the pairs of lines $\{(\alpha, \alpha'), (\beta, \beta'), (\gamma, \gamma')\}$ define on the line ε a triple of pairs $\{(A', A''), (B', B''), (C', C'')\}$ in involution w.r.t. some involution f, and the lines $\{\alpha' = AA'', \beta' = BB''\}$ intersect at a point D, such that $\{A, B, C, D\}$ are in general position. Then, define C^* to be the intersection $DC \cap \varepsilon$. By applying Desargues' involution theorem to the resulting complete quadrangle $ABCD$, we conclude that $\{(A', A''), (B', B''), (C', C^*)\}$ are in involution w.r.t. some involution g. Since the involutions $\{f, g\}$ coincide at the two pairs $\{(A, A'), (B, B')\}$, they coincide everywhere. Hence $C'' = C^*$ and the line $\gamma' = C'C''$ passes also through D. □

3 The three isogonal circles of a point

In this section we prove the *existence* of the *isogonal pencil* of three circles $\{\kappa_A, \kappa_B, \kappa_C\}$ of a point D not lying on any side-line of the triangle of reference ABC (see Figure 6). The necessary calculations involve the notion of *"cross ratio"* or *"anharmonic ratio"* (AB, CD), defined for a quadruple of collinear points $\{A, B, C, D\}$ through the ratios of their directed segments $(AB, CD) = (CA/CB) : (DA/DB)$ ([9, p. 86], [3, p. 22]). The cross ratio can also be defined for a quadruple or *"pencil"* of lines, denoted by $A(BCDE)$, which pass through the point A, called *"vertex"*, $\{B, C, D, E\}$ being points on the respective lines. The cross ratio for such a pencil, denoted by $A(BC, DE)$, is defined by intersecting the quadruple with a line ε, not passing through the vertex A and setting it equal to the cross ratio $(B'C', D'E')$ of the corresponding intersection points. This definition relies on the theorem asserting that ([9, p. 89]), *"the cross ratio of the pencil does not depend on the particular line* ε *intersecting the lines of the pencil."* By means of this, all notions relating to cross ratios of points on a line transfer to corresponding notions of pencils of four lines through a point. The notion of *"harmonic quadruple"* of four collinear points is characterized by the particular value $(AB, CD) = -1$ of their cross ratio. Four collinear points satisfying this relation are said to form *"harmonic pairs"* denoted by $(A, B) \sim (C, D)$. Use is also made of the symbol $A = B(CD)$, read *" A is harmonic conjugate to B w.r.t."* $\{C, D\}$ and meaning that $(A, B) \sim (C, D)$. Related to this is the notion of *"harmonic pencil"*, whose cross ratio is $A(BC, DE) = -1$, so that every line intersects it in a harmonic quadruple. Perhaps the most prominent harmonic pairs consist of the endpoints $\{B, C\}$ of the side of a triangle ABC and the traces $\{D, E\}$ on BC (see Figure 10) of the internal and external bisectors of $\widehat{A}$, which are diametral

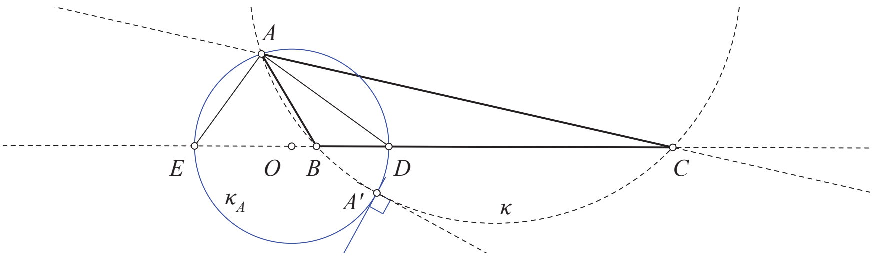

Figure 10 Apollonian circle κ_A, $(D,E)\sim(B,C)$ (Color figures online)

points of the corresponding Apollonian circle κ_A of the triangle ([5, p. 15]). The resulting pencil of lines $A(BCDE)$ being also *harmonic*.

Figure 10 suggests also another important property of harmonic pairs: *"the points of one pair are inverse w.r.t. to the circle having diametral the points of the other pair."* The term *"inverse"* appearing in this property means that the two points are related by an *"inversion transformation"* ([9, p. 144], [17, p. 75]). Such a transformation is defined by a fixed circle $\kappa(O,r)$ and associates to each point $X \neq O$ the point Y on the line OX such that the directed segments satisfy $OX \cdot OY = r^2$. Among the basic properties of these transformations are that: (i) they map circles not passing through O to circles and circles passsing through O to lines, (ii) they are *"conformal transformations"*, i.e., they preserve the angle of two intersecting circles or lines, (iii) they are involutive, i.e., they are coincident with their inverse transformations, satisfying $f^{-1}=f$ and (iv) the *"circle of inversion"* $\kappa(O,r)$ intersects orthogonally every circle passing through two inverse points $\{X, Y=f(X)\}$. In the example of Figure 10 the circle of inversion is κ_A, $C=f(B)$ are inverse points, and the circumcircle κ of ABC intersects orthogonally the circle κ_A.

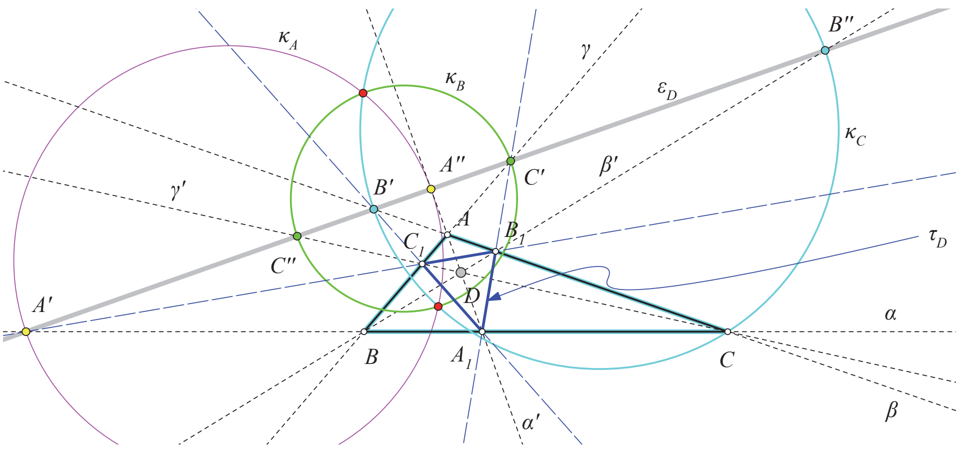

Figure 11 The isogonal pencil of D w.r.t. triangle ABC (Color figures online)

Figure 11 shows the configuration of the *isogonal pencil* of the three *trilinear isogonal circles* we are going to define. From well-known properties of complete quadrangles, as the $AB_1A_1C_1$ appearing in the figure, follows the existence of several harmonic pairs, the most important being those contained in the tripolar ε_D of D:

$$(A', A'') \sim (B', C'), \quad (B', B'') \sim (C', A'), \quad (C', C'') \sim (A', B'). \tag{2}$$

These involve the intersections $\{A' = BC \cap B_1C_1, B' = CA \cap C_1A_1, C' = AB \cap A_1B_1\}$, coinciding with the harmonic conjugates $\{A' = A_1(BC), B' = B_1(CA), C' = C_1(AB)\}$ and the intersections $\{A'' = AD \cap \varepsilon_D, B'' = BD \cap \varepsilon_D, C'' = CD \cap \varepsilon_D\}$. Points $\{A_1, B_1, C_1\}$ are the traces on the sides $\{\alpha = BC, \beta = CA, \gamma = AB\}$ of the cevians $\{\alpha', \beta', \gamma'\}$ through D, defining the cevian triangle $\tau_D = A_1B_1C_1$.

The validity of the relations in (2) results by considering pencils of lines of a complete quadrangle, known to be harmonic ([9, p. 101]), such as, f.e., the pencil of lines $A_1(A'A''BC')$ intersecting on β the harmonic pairs $(B', B_1) \sim (A, C)$ and, consequently also on ε_D the harmonic pairs $(B', C') \sim (A'', A')$. The next theorem, which is new and guarantees the existence of our configuration, is a consequence of all these facts and the following two well-known lemmata ([16, VIII,p. 35]).

Lemma 2. *The three pairs* $\{(A, A'), (B, B'), (C, C')\}$ *are in involution if and only if, the cross ratio of a quadruple of them, comprising at least one element from each pair, is equal to the cross ratio of their conjugates.*

Proof. The condition is *necessary*, since taking such a quadruple, $\{A, B, C, A'\}$ say, its conjugate $\{A', B', C', A\}$ results by an involutive transformation which preserves cross ratios. Hence

$$(A'B', C'A) = (AB, CA').$$

This condition is also *sufficient*. To see this consider the involution $g : A \mapsto A', B \mapsto B'$. It suffices to show that $g(C) = C'$. Let $g(C) = C''$. Then, by the preservation of the cross ratio

$$(A'B', C''A) = (AB, CA') = (A'B', C'A),$$

last equation being valid per assumption. But $(A'B', C''A) = (A'B', C'C) \Rightarrow C'' = C'$. Analogous proofs can be given for all quadruples like the $\{A, B, C, A'\}$ considered above. □

Lemma 3. *If the pairs* $\{(A, A'), (B, B'), (C, C')\}$ *are in involution and* $(B, C) \sim (A, A')$, *then also* $(B', C') \sim (A, A')$. *Conversely, for arbitrary pairs* $\{(A, A'), (B, B')\}$ *construct the pair of harmonic conjugates* $\{C = B(AA'), C' = B'(AA')\}$. *Then, the pairs* $\{(A, B')$, (B, B'), $(C, C')\}$ *are in involution.*

Proof. The proofs are formal applications of the rules:

$$\text{first claim:} \quad (BC, AA') = -1 \quad \Rightarrow \quad (B'C', A'A) = -1.$$
$$\text{converse:} \quad -1 = (BC, AA') = (B'C', AA') = (B'C', A'A).$$

In the converse, the first two equalities are assumed and the last is a symmetry property of the harmonic quadruple. The proof results from the last equalities, implying $(BC, AA') = (B'C', A'A)$, and applying Lemma 2. □

Theorem 1. *With the preceding notation and conventions, the three circles* $\{\kappa_A, \kappa_B, \kappa_C\}$ *on respective diameters* $\{A'A'', B'B'', C'C''\}$ *form a pencil of intersecting type and the angles at their intersection points measure* $60°$.

Proof. The fact that the circles form a pencil follows Lemma 1, guaranteeing that the pairs $\{(A'A''), (B'B''), (C'C'')\}$ are in involution, and from remark 1. That the kind of the pencil is the claimed one, follows from the aforementioned *harmonicity property* of the pairs $(A', A'') \sim (B', C')$ and Lemma 3, which implies that also $(A', A'') \sim (B'', C'')$. This shows that the inversion on circle κ_A maps κ_B onto κ_C and analogous properties for the cyclic permutations of the letters. Thus, the three circles intersect and, by the conformality property of inversions, the angles around the intersection points of the three circles must be equal, thereby completing the proof. □

4 Points contained in their radical axis

The location of the *trilinear radical* of D, defined as the radical axis of the three *isogonal cirlces*, whose existence has been established in the previous section, can be determined through its intersection D_0 with the trilinear polar ε_D (see Figure 12). In fact, using

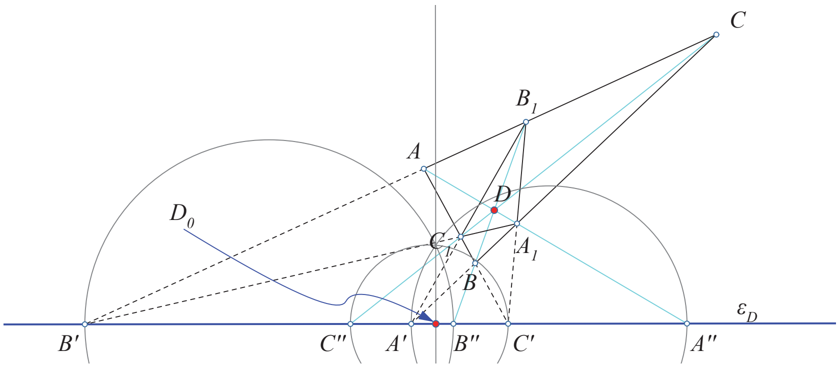

Figure 12 Locating the trilinear radical of D (Color figures online)

the involution f, interchanging the diametral points on ε_D of the member-circles of the pencil, point D_0 is interchanged via f with the point at infinity P_∞ of ε_D. Geometrically this amounts to identifying the radical axis with the, so to say *"biggest"* circle of the pencil.

For this, we can select the first two basic points of the homogeneous system on ε_D to be the points $\{A', B'\}$. Every point of ε_D is then expressed as a linear combination $X = xA' + yB'$, and the involution f is represented by a function of the form:

$$t' = f(t) = \frac{at+b}{ct-a}, \quad \text{with} \quad a^2 + bc \neq 0. \tag{3}$$

The variable t in equation (3) is the ratio $t = x/y$ and we have to find the constants $\{a, b, c\}$ from the given data.

We do this using *barycentric coordinates* w.r. to the triangle of reference ABC ([21]). This is a system of *"projective coordinates"* of the plane adapted to the triangle ABC and its centroid G, which, to these particular four points associates the coordinates $\{A(1,0,0), B(0,1,0), C(0,0,1), G(1,1,1)\}$ describing every other point of the plane P by its *"coordinates"* (x,y,z) defined up to non-zero multiplicative constant and representing it also formally as a linear combination $P = xA + yB + zC$. Coordinates restricted to satisfy $x + y + z = 1$ are called *"absolute barycentrics"*.
Identifying points D with their coordinates (u,v,w) and referring to Figure 12, we find by standard calculations in barycentrics ([21, p. 25]) that:

$$\begin{array}{lll} A' = (0 : v : -w), & B' = (-u : 0 : w), & C' = (u : -v : 0), \\ A'' = (2u : -v : -w), & B'' = (-u : 2v : -w), & C'' = (-u : -v : 2w). \end{array} \tag{4}$$

In our base $\{A', B'\}$ of ε_D we have the representation of the points:

$$\begin{array}{ll} A' = 1 \cdot A' + 0 \cdot B', & A'' = A' + 2B', \\ B' = 0 \cdot A' + 1 \cdot B', & B'' = 2A' + B'. \end{array}$$

As we noticed in Section 2, the involution f is determined by knowing two points and their images, which is the case with $\{A', B'\}$ and $\{A'', B''\}$. Writing explicitly the ratios

$$\frac{x'}{y'} = \frac{a(x/y) + b}{c(x/y) - a} = \frac{ax + by}{cx - ay},$$

we come to the equivalent equation, which can be used to find the constants $\{a, b, c\}$:

$$c(xx') - b(yy') = a(xy' + x'y).$$

Substitution in this of the coordinates for $\{A'(1,0), A''(1,2)\}$ and $\{B'(0,1), B''(2,1)\}$ leads respectively to $\{c = 2a,\ b = -2a\}$ and the form of the involution:

$$t' = f(t) = \frac{t-2}{2t-1}. \tag{5}$$

On the other side, the point at infinity of the trilinear polar ε_D of D, which has the coefficients $(1/u : 1/v : 1/w)$, is calculated as intersection with the line at infinity, with coefficients $(1 : 1 : 1)$, producing by the vector product, up to a factor, the point at infinity:

$$\begin{aligned} P_\infty &= (u(w - v) : v(u - w) : w(v - u)) \\ &= (2v - u - w)A' + (-2u + v + w)B' = xA' + yB'. \end{aligned} \tag{6}$$

By the involutive nature of $f(t)$, considering the point $D_0 = x'A' + y'B'$ and setting in equation (5) $x'/y' = t' = f(x/y)$, for $\{x, y\}$ given by equation (6), produces after a short calculation the coordinates:

$$D_0(u,v,w) = \begin{pmatrix} u(v + w - 2u) \\ v(w + u - 2v) \\ w(u + v - 2w) \end{pmatrix} = \begin{pmatrix} u(1 - 3u) \\ v(1 - 3v) \\ w(1 - 3w) \end{pmatrix}, \tag{7}$$

the last equation being valid only if $u+v+w = 1$, i.e., when we use *absolute* barycentrics.

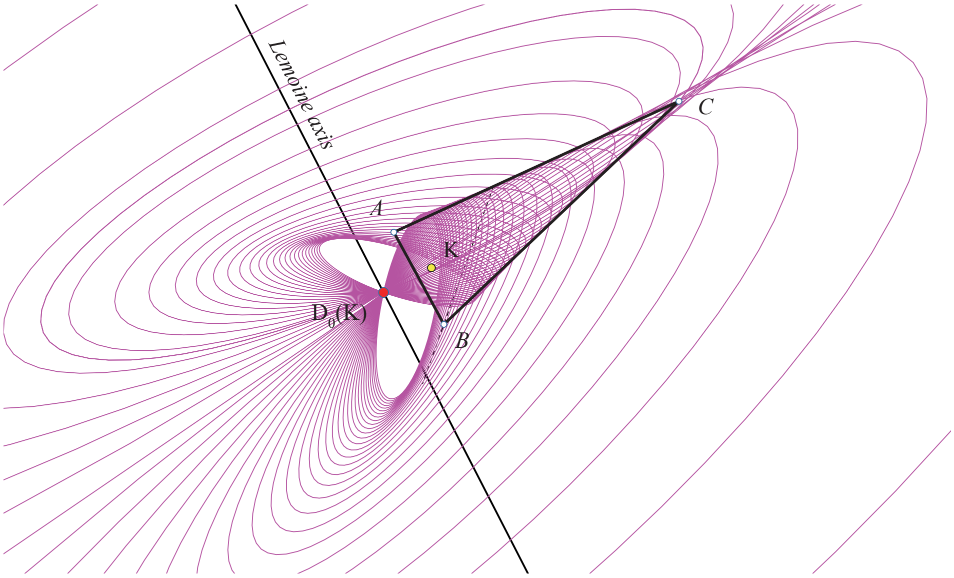

Figure 13 The quadratic transformation $D \mapsto D_0$ (Color figures online)

Remark 2. The transformation $D \mapsto D_0$, described by equations (7), is a so-called *"Cremona quadratic"* transformation of the projective plane ([19, p. 19]), mapping lines to conics. Figure 13 illustrates the behavior of this transformation showing the conics which are transforms of various lines through the symmedian point K of the triangle ABC. This could be used to give an alternative proof for the uniqueness of K, as claimed below, in Theorem 3. Since the lines through K map to conics through $D_0(K)$, and the map restricted to such a line is not periodic, no other point of the line can map to $D_0(K)$.

The next theorem formulates the main result of this section (see Figure 14).

Theorem 2. *The points D of the plane, which have the property to lie on their trilinear radical η_D w.r. to the triangle ABC, lie on an algebraic curve of fifth degree (a binary quintic), having double points at the vertices and the centroid of ABC and passing through the middles of the sides of the triangle. Further, the tangents to the curve branches at the centroid G and the vertices are orthogonal, in the case of vertices coinciding with the bisectors of the respective angles of ABC.*

Proof. This results by writing the condition of orthogonality of the two lines $\{\varepsilon_D, DD_0\}$ ([21, p. 54]):

$$S_A pp' + S_B qq' + S_C rr' = 0. \tag{8}$$

Here $\{(p : q : r), (p' : q' : r')\}$ represent the points at infinity of the two lines, the symbols $\{S_A, S_B, S_C\}$ being defined by the expressions:

$$S_A = (b^2 + c^2 - a^2)/2, \qquad S_B = (c^2 + a^2 - b^2)/2, \qquad S_C = (a^2 + b^2 - c^2)/2,$$

in which $\{a = |BC|, b = |CA|, c = |AB|\}$ are the side-lengths of the triangle of reference ABC. The point at infinity of ε_D is given by equation (6) and this of DD_0 is easily found using equation (7):

$$(u^2(v+w) - u(v^2+w^2) \ : \ v^2(w+u) - v(w^2+u^2) \ : \ w^2(u+v) - w(u^2+v^2)).$$

Introducing these into equation (8) leads to the desired equation of fifth degree:

$$\begin{aligned} &S_A u^2(w-v)(u(v+w) - (v^2+w^2)) \ + \ \ldots \ = \ 0, \quad \Leftrightarrow \\ &a^2 vw(v-w)(vw-u^2) \ + \ \ldots \ = \ 0. \end{aligned} \tag{9}$$

where the dots, as usual in calculations with barycentrics, denote the sum of the other two expressions resulting from the first by cyclic permutation of the letters. Figure 14 shows

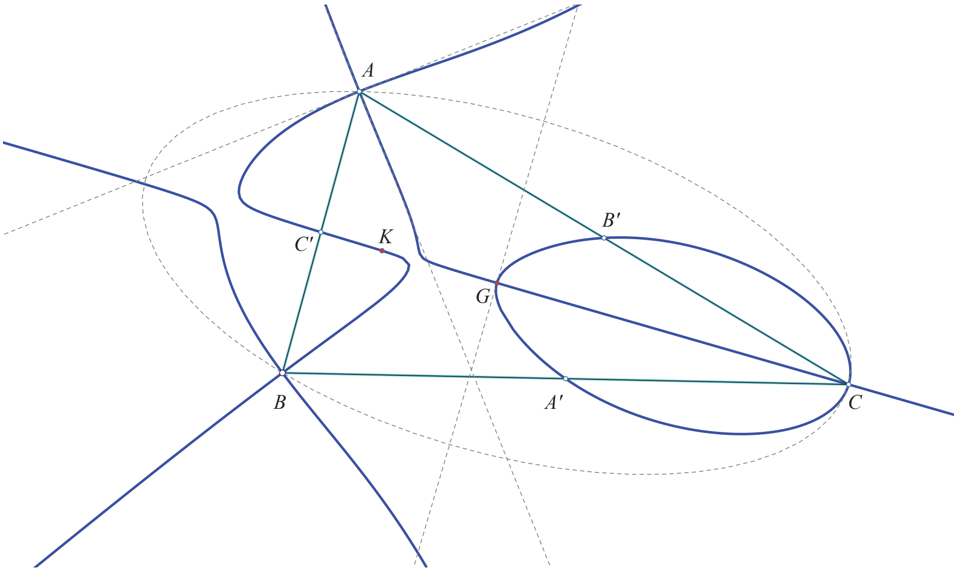

Figure 14 Curve of points $\{D\}$ having $DD_0 \perp \varepsilon_D$ (Color figures online)

such a *“quintic”* and the tangents of the two branches at A, coinciding with the bisectors of the angle $\widehat{A}$.

Actually we could stop the proof here, since equation (9) identifies the curve with *“Stother’s quintic”* ([12], [11, p. 219]) having all the stated properties and many more. Since however explicit proofs of the orthogonality of the tangents at the nodes, especially at G, are to the best of my knowledge not readily available, I sketch them briefly.

The fact that the curve passes through the vertices, the middles of the sides, the centroid G and the symmedian point K is easily verified by replacing in equation (9) the coordinates of these points $\{(1:0:0), (0:1:0), \ldots, (1:1:1), (a^2, b^2, c^2)\}$.

To show that the curve has at the vertices and the centroid double points or *“nodes”* we apply the standard procedure of algebraic geometry ([13]) for the location and the kind of *"singular points"*, involving the partial derivatives of f, which is the function on the left side of equation (9). The *"singular"* points are those at which all first order derivatives vanish. At each one of these points is then calculated the 3×3 matrix of second derivatives

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial u^2} & \frac{\partial^2 f}{\partial v \partial u} & \frac{\partial^2 f}{\partial w \partial u} \\ \frac{\partial^2 f}{\partial u \partial v} & \frac{\partial^2 f}{\partial v^2} & \frac{\partial^2 f}{\partial w \partial v} \\ \frac{\partial^2 f}{\partial u \partial w} & \frac{\partial^2 f}{\partial v \partial w} & \frac{\partial^2 f}{\partial w^2} \end{pmatrix},$$

the so-called *"Hessian"* of f. The matrix, because of the homogeneity of the function f, is always singular, i.e., has a vanishing determinant and in the case of nodes it is non-zero. This implies that the conic represented by the matrix is degenerate, consisting of the product of two lines, which are precisely the *"nodal tangents"* of the curve at the singular point.

From the partial derivatives of first order in our case, characteristically one is:

$$\begin{aligned}\frac{\partial f}{\partial u} = S_A[u(v-w)(2(v^2+w^2) - 3u(v+w))] \\ + S_B[v^2(2u(v-u) - (u-w)^2)] \\ + S_C[w^2(2u(u-w) + (u-v)^2)],\end{aligned}$$

analogous expressions being valid for the other partial derivatives of first order. Replacing in them the coordinates of the four points $\{A, B, C, G\}$ we find that they satisfy

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial w} = 0,$$

Hence these points are *singular*. To further inspect the kind of the singularity we consider the partial derivatives of the second order, characteristically one of them being:

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2} = 2S_A[(v-w)((v^2+w^2) - 3u(v+w))] \\ + 2S_B[v^2(v+w-3u)] \\ - 2S_C[w^2(v+w-3u)].\end{aligned}$$

Calculating the other derivatives at the points $\{A, B, C, G\}$ we find indeed that the corresponding *Hessian* matrix is non-vanishing but degenerate. Characteristically, the matrix at $A(1, 0, 0)$ being equal to

$$H_f(1,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2c^2 & 0 \\ 0 & 0 & 2b^2 \end{pmatrix}.$$

The corresponding conic, traditionally called *"tangent cone"* at the singularity point is in this case

$$-2c^2v^2 + 2b^2w^2 = 0 \quad \Leftrightarrow \quad (cv - bw)(cv + bw) = 0,$$

The two lines $\{cv - bw = 0,\ \ cv + bw = 0\}$ representing the bisectors of the angle $\widehat{A}$. Hence the proof concerning the claim for the vertex A of the triangle. Analogous arguments prove also the claim for the other vertices. At the centroid G the matrix is seen to be equal to the degenerate:

$$H_f(1,1,1) \;=\; 2\begin{pmatrix} b^2-c^2 & a^2-b^2 & c^2-a^2 \\ a^2-b^2 & c^2-a^2 & b^2-c^2 \\ c^2-a^2 & b^2-c^2 & a^2-b^2 \end{pmatrix}.$$

Disregarding the factor, if the corresponding quadratic form is the product of two lines

$$(pu + qv + rw)\cdot(p'u + q'v + r'w),$$

then the coefficients must satisfy the relations

$$\begin{array}{lll} pp' = \mu(b^2-c^2), & qq' = \mu(c^2-a^2), & rr' = \mu(a^2-b^2), \\ qr' = rq' = \mu(b^2-c^2), & rp' = r'p = \mu(c^2-a^2), & pq' = p'q = \mu(a^2-b^2), \end{array} \tag{10}$$

for some constant μ. On the other side, the orthogonality of these two lines expressed in barycentrics is given by ([15, II,p. 41] gives the corresponding condition for *trilinears*):

$$a^2pp' + b^2qq' + c^2rr' - S_A(qr'+q'r) - S_B(rp'+r'p) - S_C(pq'+p'q) = 0. \tag{11}$$

A short computation shows that latter equation is indeed satisfied if we replace in it the coefficients with their equivalents from equations (10), thereby proving the orthogonality of the tangents at G and completing the proof of the theorem. □

Remark 3. It is known that the two tangents of the curve at G are identical with the *axes* of the *"Steiner ellipse"* ([10, p. 108], [7, p. 378]) of the triangle ABC, seen in Figure 14.

5 Trilinear isodynamics of the symmedian point

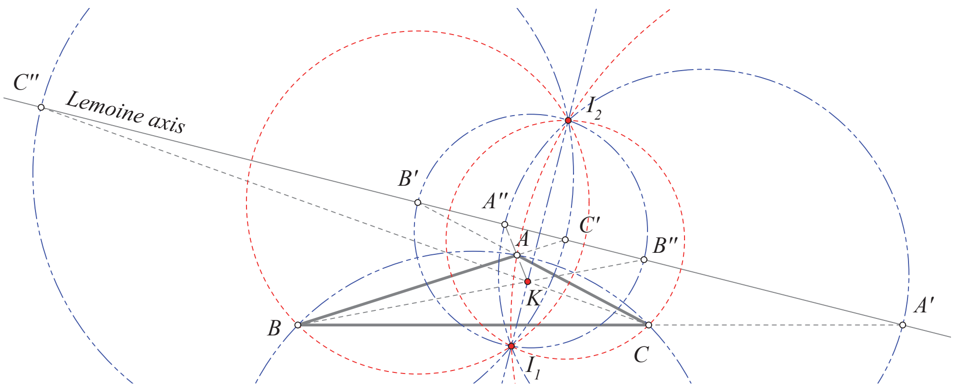

Figure 15 Coincidence at K of two *"isodynamic-points"* notions (Color figures online)

Theorem 3. *With the notation and definitions adopted so far, the symmedian point K is the only point of the plane whose trilinear isodynamics coincide with the traditional isodynamic points $\{I_1, I_2\}$ of the triangle of reference ABC.*

Proof. If for a point D the trilinear isodynamics coincide with the traditional isodynamic points of the triangle, then the line of centers of the trilinear isogonal circles of D must coincide with the *"Lemoine axis"* of ABC. This identifies the trilinear polar of D with the Lemoine axis, hence also D with the symmedian point K of ABC. This proves the uniqueness part of the theorem. Figure 15 illustrates the case showing also the two pencils of circles: the *trilinear isogonal circles of* K with diameters $\{A'A'', B'B'', C'C''\}$ and the *Apollonian circles of* ABC, whose centers are the points $\{A', B', C'\}$.

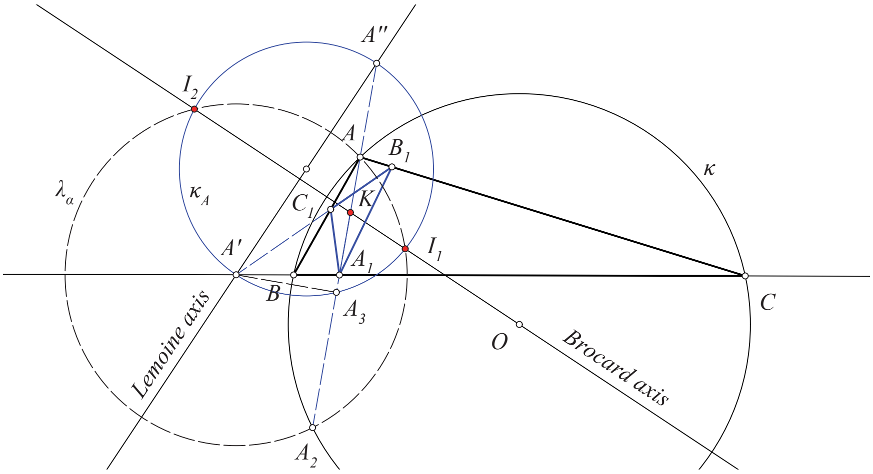

Figure 16 Circle κ_A passing through $\{I_1, I_2\}$ (Color figures online)

Figure 16 completes the proof of the other half of the theorem, showing why the circle κ_A on diameter $A'A''$ passes through the traditional isodynamic points $\{I_1, I_2\}$. In fact, it is known that the Lemoine axis is the polar of K w.r. to the circumcircle $\kappa(O)$ of ABC and the isodynamic points (I_1, I_2) are harmonic conjugate to (K, O) ([14, p. 295]). This implies that the polar of A' passes through K. Since it is also orthogonal to $A'O$, it coincides with the orthogonal to $A'O$ from K, which is the line KA''. By the general properties of polars, points A' and $A_3 = KA'' \cap OA'$ are inverse w.r. to κ_A, hence latter circle is orthogonal to κ. Since it is also orthogonal to the Lemoine axis, the circle κ_A belongs to the pencil $\mathcal{D}'$ of circles that are orthogonal to the pencil $\mathcal{D}$ generated by κ and the Lemoine axis. Hence it passes through $\{I_1, I_2\}$. Analogous is the proof also for the other two circles $\{\kappa_B, \kappa_C\}$ on diametters correspondingly $\{B'B'', C'C''\}$. $\square$

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