Πανεπιστήμιο Κρήτης
Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών

Κυματική Διάδοση

4η διάλεξη

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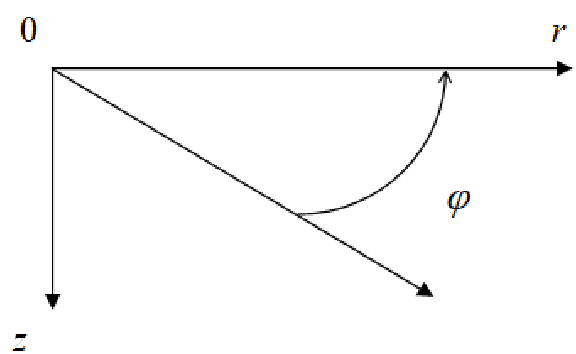
Κυλινδρικό Σύστημα

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0$$

Εξίσωση Helmholtz
(αρμονικά κύματα)

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$



$$x = r\cos\varphi,\ y = r\sin\varphi,\ z = z$$

Κυλινδρικό Σύστημα

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2} + \frac{\partial^2 p}{\partial z^2} + k^2 p = 0$$

$$p(r,z,\varphi) = F(r,\varphi)u(z)$$

Κυλινδρικό Σύστημα

$$p(r,z,\varphi) = F(r,\varphi)u(z)$$

$$\frac{1}{rF}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r}) + \frac{1}{r^2 F}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{u}\frac{d^2 u}{dz^2} + k^2 = 0$$

$$\frac{1}{u}\frac{d^2 u}{dz^2} + k_z^2 = 0$$

$$\frac{1}{rF}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r}) + \frac{1}{r^2 F}\frac{\partial^2 F}{\partial \varphi^2} + q^2 = 0$$

$$k^2 = k_z^2 + q^2$$

Κυλινδρικό Σύστημα

$$\frac{1}{u}\frac{d^2u}{dz^2}+k_z^2=0$$

$$u(z)=D_1e^{ik_zz}+D_2e^{-ik_zz}$$

$$F(r,\varphi)=R(r)\Phi(\varphi)$$

$$\frac{1}{R}(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr})+\frac{1}{r^2\Phi}\frac{d^2\Phi}{d\varphi^2}=-q^2$$

Κυλινδρικό Σύστημα

$$\frac{1}{rF}\frac{\partial}{\partial r}(r\frac{\partial F}{\partial r})+\frac{1}{r^2F}\frac{\partial^2 F}{\partial \varphi^2}+q^2=0$$

$$F(r,\varphi)=R(r)\Phi(\varphi)$$

$$\frac{r^2}{R}(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr})+q^2r^2=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$$

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi \qquad \frac{1}{R}(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr})+(q^2-\frac{m^2}{r^2})=0$$

Κυλινδρικό Σύστημα

$$\frac{d^2\Phi}{d\varphi^2} = -m^2\Phi$$

$$\Phi = E_1 e^{im\varphi} + E_2 e^{-im\varphi}$$

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \left(q^2 - \frac{m^2}{r^2}\right) = 0$$

Εξίσωση Bessel

Η Εξίσωση Bessel

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\left(q^2-\frac{m^2}{r^2}\right)=0$$

$$\left[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(x^2-\nu^2)\right]f(x)=0$$

Εξίσωση Bessel ν τάξης

Λύσεις	$J_\nu(x)$	$N_\nu(x)$	Neumann
	$H_\nu^{(1)}(x)$	$H_\nu^{(2)}(x)$	Hankel

Η εξίσωση Bessel

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} (\frac{x}{2})^{2n+\nu}$$

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(-\nu+n+1)} (\frac{x}{2})^{2n-\nu}$$

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} (\frac{x}{2})^{2n+m}$$

Για m ακέραιο

Συναρτήσεις Neumann

$$N_m(x) = \frac{1}{\pi}\left\{\frac{\partial J_m}{\partial m} - (-1)^m \left(\frac{\partial J_{-m}}{\partial m}\right)\right\}$$

Η εξίσωση Bessel

Συναρτήσεις Hankel

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

$$\text{για } x \to 0 \quad \begin{cases} J_\nu(x) \to \dfrac{1}{\Gamma(\nu+1)}\left(\dfrac{x}{2}\right)^\nu & \nu \geq 0 \\ N_\nu(x) \to \begin{Bmatrix} \dfrac{2}{\pi}[\ln(\dfrac{x}{2}) + \gamma], & \nu = 0 \\ -\dfrac{\Gamma(\nu)}{\pi}(\dfrac{2}{x})^\nu, & \nu > 0 \end{Bmatrix} \end{cases}$$

$$\gamma = -\int_0^\infty e^{-t} \ln t\, dt$$

Η εξίσωση Bessel

$$\text{για} \quad x >> \quad \begin{cases} J_\nu(x) \rightarrow \sqrt{\dfrac{2}{\pi x}} \cos(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}) \\ N_\nu(x) \rightarrow \sqrt{\dfrac{2}{\pi x}} \sin(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4}) \\ H_\nu^{(1)}(x) \rightarrow \sqrt{\dfrac{2}{\pi x}} \exp\{i(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4})\} \\ H_\nu^{(2)}(x) \rightarrow \sqrt{\dfrac{2}{\pi x}} \exp\{-i(x - \dfrac{\nu\pi}{2} - \dfrac{\pi}{4})\} \end{cases}$$

Η εξίσωση Bessel

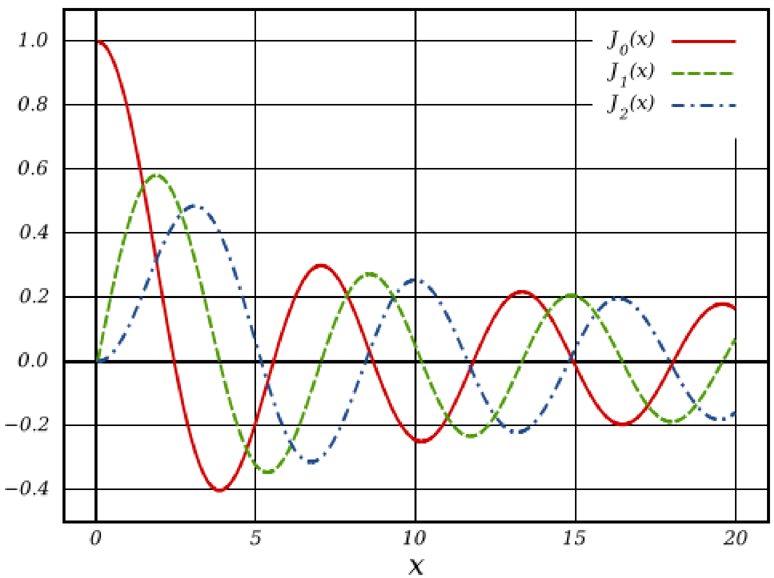
$$f_{\nu-1}(x) + f_{\nu+1}(x) = \frac{2\nu}{x} f_{\nu}(x)$$

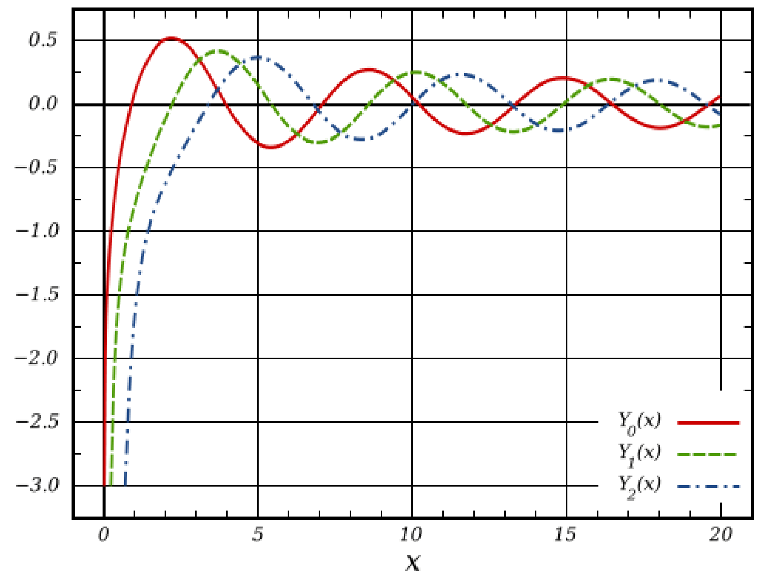
$$f_{\nu-1}(x) - f_{\nu+1}(x) = 2\frac{df_{\nu}(x)}{dx}$$

$$\frac{\nu}{x} f_{\nu}(x) - f_{\nu+1}(x) = \frac{df_{\nu}(x)}{dx}$$

$$f_{\nu-1}(x) - \frac{\nu}{x} f_{\nu}(x) = \frac{df_{\nu}(x)}{dx}$$

Η εξίσωση Bessel





$$N_\nu(x) \to Y_\nu(x)$$

Η εξίσωση Bessel

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\left(q^2-\frac{m^2}{r^2}\right)=0$$

$$\left[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(x^2-\nu^2)\right]f(x)=0$$

$$\left[\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}\right)+\left((qr)^2-m^2\right)\right]R=0$$

$$R(r)=E_3J_m(qr)+E_4N_m(qr)$$

$$R(r)=E_5H_m^{(1)}(qr)+E_6H_m^{(2)}(qr)$$

Η εξίσωση Bessel

$$R(r) = E_5 H_m^{(1)}(qr) + E_6 H_m^{(2)}(qr)$$

$qr \gg$

$$H_m^{(1)}(qr) \to \sqrt{\frac{2}{\pi qr}} \exp\{i(qr - \frac{m\pi}{2} - \frac{\pi}{4})\}$$

$$H_0^{(1)}(qr) \to \sqrt{\frac{2}{\pi qr}} \exp\{i(qr - \frac{\pi}{4})\}$$

Κυματική εξίσωση στο κυλινδρικό σύστημα συντεταγμένων

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r})+\frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2}+\frac{\partial^2 p}{\partial z^2}+k^2 p=0$$

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$$R(r)=E_5 H_m^{(1)}(qr)+E_6 H_m^{(2)}(qr)$$

$$p(r,z,\varphi)=R(r)\Phi(\varphi)u(z)$$

$$p(x,y,z)=p_x(x)p_y(y)p_z(z)$$

Ίδιο πρόβλημα

Κυματική εξίσωση στο κυλινδρικό σύστημα συντεταγμένων

Αξονική συμμετρία

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\frac{\partial^2 p}{\partial z^2}+k^2 p=0$$

$$\Downarrow$$

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{\partial^2 p}{\partial z^2}+k^2 p=0$$