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The problem of Dirichlet for evolution one-dimensional p-Laplacian with nonlinear source

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Abstract

In the present paper we consider the Dirichlet problem for one-dimensional p-Laplacian with nonlinear source. We obtain new a priori estimates of a solution and of the gradient of a solution and formulate conditions guaranteeing the global solvability of this problem. Our consideration includes singular case as well.

Keywords: Degenerate quasilinear parabolic equations; Singular quasilinear parabolic equations

0. Introduction and main results

In the present paper we consider the following quasilinear parabolic equation:

$$u_t = \left(|u_x|^{p-2}u_x\right)_x + \lambda g(u) \quad \text{in } Q_T = (0,T)\times(-l,l), \tag{0.1}$$

where $p > 1$, λ are constants, coupled with homogeneous Dirichlet boundary condition

$$u(t,\pm l) = 0 \quad \text{for } t \in [0,T] \tag{0.2}$$

and initial condition

$$u(0,x) = u_0(x) \quad \text{for } x \in [-l,l]. \tag{0.3}$$

We assume that

$$g(u) = |u|^{q-1}u, \quad q \geqslant 1 \quad \text{or} \quad g(u) = |u|^q, \quad q \geqslant 0 \quad \text{or} \quad g(u) = u^q, \quad q \geqslant 0 \quad \text{if defined.} \tag{0.4}$$

For $p > 2$ Eq. (0.1) is degenerate and for $p \in (1,2)$ is singular.

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Definition. We say that $u(t,x)$ is a global generalized solution of problem (0.1)–(0.3) if $u_x(t,x)$ is Hölder continuous function, $u_t(t,x) \in L_2(0,T;H^{-1}(-l,l))$ and

$$\int_0^T \langle u_t, \phi\rangle\,dt + \int_{Q_T} |u_x|^{p-2}u_x\phi_x\,dx\,dt = \int_{Q_T} \lambda g(u)\phi\,dx\,dt, \quad \forall\phi(t,x) \in L_2\big(0,T;\mathring{H}^1(-l,l)\big).$$

Conditions (0.2), (0.3) are satisfied in the classical sense.

Here $\langle\,,\rangle$ denotes the pairing between $H^{-1}(-l,l)$ and $\mathring{H}^1(-l,l)$.

The local solvability of problem (0.1)–(0.3) follows from [9]. Also from [9] it follows that if $q < p-1$, then there exists a global solution, for the critical case $q = p-1$ the global solution exists if the measure of the domain is sufficiently small, otherwise there is no global solution. For $q > p-1$ the blow up of the solution was demonstrated. In the present paper for the one-dimensional case we formulate a general condition (see (0.6)) guaranteeing the global solvability of problem (0.1)–(0.3). If $q < p-1$ this condition is fulfilled with arbitrary initial function and domain, if $q = p-1$ this condition is fulfilled with arbitrary initial function if the size of the domain is small (see (0.9)). Finally if $q > p-1$ this condition becomes the smallness restriction connecting the size of the domain, the initial function, and parameters λ, p, q. The proposed condition is given in the explicit form and is easily verifiable. Moreover the estimates of u and u_x are also given in an explicit form. For more details see Examples 1–4 below.

Let us pass to the formulation of the result.

Suppose that the initial function $u_0(x)$ satisfies the following conditions:

$$u_0(x) \in C^{1+\alpha}\big([-l,l]\big), \quad u_0(\pm l) = 0, \quad \big|u_0'(x)\big| \leqslant K. \tag{0.5}$$

Assume that there exists a positive constant M such that

$$M \geqslant l_* K \quad \text{and} \quad |\lambda| M^q < (p-1) l_*^{1-p} M^{p-1}, \tag{0.6}$$

where

$$l_* = \frac{3l^2 + 2l}{2}.$$

Below we will give several examples concerning condition (0.6).

Theorem 1. *Suppose that conditions* (0.4)–(0.6) *are fulfilled. Then there exists a global generalized solution of problem* (0.1)–(0.3) *such that*

$$\max_{Q_T} |u| \leqslant M, \tag{0.7}$$

$$\max_{Q_T} |u_x| \leqslant (1+2l)\max\left\{K, \frac{4l+2}{3l^2+2l}M, \left(\frac{|\lambda| M^q}{(p-1)}\right)^{\frac{1}{p-1}}\right\}. \tag{0.8}$$

If, in addition, $g(u)$ is Lipschitz continuous function on $[-M,M]$, then the solution is unique.

Remark 1. Estimates (0.7) and (0.8) are independent of T.

Example 1. If $q < p-1$, then for any $p > 1$ we can always find a (sufficiently big) positive constant M such that

$$|\lambda| M^q < (p-1) l_*^{1-p} M^{p-1}.$$

Thus for such q Theorem 1 guarantees the existence of a global generalized solution of problem (0.1)–(0.3) satisfying (0.7) and (0.8).

Note that in [8] for $p > 2$ it was shown that for nonnegative initial data there exists a global nonnegative solution if $q < p-1$. For $q > p-1$ the existence of a global solution was proved under additional assumption on the smallness of the initial data and for sufficiently large (nonnegative) initial data it was shown that the solution blows up in a finite time. The blow-up results for $q > p-1$, $p > 2$ were also proved in [3].

Example 2. Consider the critical case $q = p - 1$. Condition (0.6) takes the form

$$M \geqslant l_* K, \quad |\lambda| < \frac{q}{l_*^q}.$$

Put $M = Kl_*$ and rewrite the second inequality as follows

$$|\lambda|\left(3l^2 + 2l\right)^q < q2^q \quad \text{or} \quad 3l^2 + 2l < 2\left(\frac{q}{|\lambda|}\right)^{\frac{1}{q}}. \tag{0.9}$$

Thus if (0.9) is fulfilled, then Theorem 1 guarantees the existence of a global generalized solution of problem (0.1)–(0.3) for $q = p - 1$ for any $p > 1$. Moreover for this solution we have

$$\max_{Q_T} |u| \leqslant \frac{3l^2 + 2l}{2} K < \left(\frac{q}{|\lambda|}\right)^{\frac{1}{q}} K,$$

$$\max_{Q_T} |u_x| \leqslant (1 + 2l) K \max\left\{1 + 2l, \frac{3l^2 + 2l}{2}\left(\frac{|\lambda|}{q}\right)^{\frac{1}{q}}\right\}.$$

In [4] the critical case $q = p - 1$ under the assumption $p > 2$ and $g(u) = |u|^{q-1}u$ was also considered. It was shown that if $\lambda > \lambda_1$, there are no global weak solutions, and if $\lambda \leqslant \lambda_1$, all weak solutions are global. Here λ_1 is the first eigenvalue of the problem

$$-\left(|\psi_x|^{p-2}\psi_x\right)_x = \lambda|\psi|^{p-2}\psi \quad \text{in } (-l, l), \quad \psi(\pm l) = 0.$$

Example 3. Consider equation

$$u_t = \left(|u_x|^{p-2}u_x\right) + \lambda u^{2(p-1)}. \tag{0.10}$$

Condition (0.6) takes the form

$$\frac{3l^2 + 2l}{2} K \leqslant M < \left(\frac{p-1}{|\lambda|}\right)^{\frac{1}{p-1}} \frac{2}{3l^2 + 2l}. \tag{0.11}$$

In order to find M satisfying condition (0.11), we need to impose the following restriction

$$\frac{3l^2 + 2l}{2} K < \left(\frac{p-1}{|\lambda|}\right)^{\frac{1}{p-1}} \frac{2}{3l^2 + 2l}$$

or

$$\left(3l^2 + 2l\right)^2 K < 4\left(\frac{p-1}{|\lambda|}\right)^{\frac{1}{p-1}}. \tag{0.12}$$

Hence if (0.12) is fulfilled, then for any $p > 1$ Theorem 1 guarantees the existence of a global generalized solution of problem (0.10), (0.2), (0.3) satisfying (0.7), (0.8).

Example 4. Finally, let us consider the case $p = 2$:

$$u_t = u_{xx} + \lambda g(u). \tag{0.13}$$

Condition (0.6) takes the form

$$\frac{3l^2 + 2l}{2} K \leqslant M < \left(\frac{2}{|\lambda|(3l^2 + 2l)}\right)^{\frac{1}{q-1}}.$$

In order to find M satisfying this condition, we need to impose the following restriction:

$$|\lambda|\left(3l^2 + 2l\right)^q K^{q-1} < 2^q. \tag{0.14}$$

Hence if the smallness condition (0.14) is fulfilled, then Theorem 1 guarantees the existence of a classical solution of problem (0.13), (0.2), (0.3) satisfying (0.7) and (0.8).

When $p = 2$ the blow-up properties of Eq. (0.13) have been intensively investigated by many researchers, see, for example, the survey paper [1]. It is well known that different smallness conditions on the data of problem (0.13), (0.2), (0.3) guarantee the global solvability of this problem. To the best of our knowledge smallness condition (0.14) is new.

The paper consists of two sections. In the first section we obtain a priori estimates for the regularized problem and in the second one based on these a priori estimates we prove Theorem 1.

1. A priori estimates for the regularized problem

Consider the regularized equation

$$u_{\varepsilon t} = \left(\left(u_{\varepsilon x}^{\alpha} + \varepsilon\right)^{\frac{p-2}{\alpha}} u_{\varepsilon x}\right)_x + \lambda g_M(u_\varepsilon). \tag{1.1}$$

Here $\varepsilon > 0$ is a constant and the function g_M is defined by the following:

$$g_M(\xi) = \begin{cases} g(\xi), & \text{for } |\xi| \leqslant M, \\ g(M), & \text{for } \xi > M, \\ g(-M), & \text{for } \xi < -M. \end{cases} \tag{1.2}$$

Obviously from (1.2) and (0.4) we have $-g(M) \leqslant g_M(u_\varepsilon) \leqslant g(M)$.

Concerning constants α and ε we consider three cases:

(i) if $p \geqslant 3$ we take $\alpha = 2$ and arbitrary $\varepsilon > 0$,

(ii) if $2 \leqslant p < 3$ we put $\alpha = r/m$ with r, m positive integers, $r < m$ and r even, for example, $\alpha = 2/3$, here $\varepsilon > 0$ is also arbitrary,

(iii) if $p \in (1, 2)$, then additionally to assumption (ii) we require

$$\alpha > p - 1 \quad \text{and} \quad 0 < \varepsilon \leqslant \left(\alpha - (p-1)\right)\left(\frac{M}{l_*}\right)^{\alpha}.$$

For example, if $p = \frac{3}{2}$ one can put $\alpha = \frac{2}{3}$, if $p = \frac{5}{3}$ one can put $\alpha = \frac{4}{5}$. The choice of α and ε is motivated by two reasons, the first is that for such α,

$$\left(u_x^{\alpha}\right)^{\frac{p-2}{\alpha}} = |u_x|^{p-2}$$

and the second is that $E'(\varepsilon) \geqslant 0$, where the function $E(\varepsilon)$ is defined below (see (1.5)).

The first step (Lemma 1) is to obtain the estimate $|u(t,x)| \leqslant M$ for the solution of problem (1.1), (0.2), (0.3). After this in (1.1) instead of $g_M(u_\varepsilon)$ we can take $g(u_\varepsilon)$ (due to (1.2)). The second step (Lemma 2) is to obtain the gradient estimate.

In order to simplify the notation, below in this section we will omit the subscript ε in u_ε.

Lemma 1. *If* (0.4)–(0.6) *are fulfilled, then for any classical solution of problem* (1.1), (0.2), (0.3) *the following estimate is valid*:

$$\left|u(t,x)\right| \leqslant M.$$

Proof. Rewrite Eq. (1.1) in nondivergent form

$$u_t = a_\varepsilon(u_x)u_{xx} + \lambda g_M(u),$$

where

$$a_\varepsilon(z) = \left(z^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}-1}\left((p-1)z^\alpha + \varepsilon\right), \quad a_\varepsilon(z) = a_\varepsilon(-z).$$

Define the function $h(x)$,

$$h(x) = \widetilde{M}\left(\frac{l^2 - x^2}{2} + (1+l)(l+x)\right),$$

where

$$\widetilde{M} = \frac{M}{l^*}.$$

Obviously $h'(x) \geqslant \widetilde{M}$, $h''(x) = -\widetilde{M}$. For

$$Lu \equiv u_t - a_\varepsilon(u_x)u_{xx}$$

we have

$$Lu = \lambda g_M(u) \tag{1.3}$$

and

$$Lh = h_t - a_\varepsilon(h_x)h_{xx} = \left(h'^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}-1}\left((p-1)h'^\alpha + \varepsilon\right)\widetilde{M}. \tag{1.4}$$

Consider $E(\varepsilon)$,

$$E(\varepsilon) \equiv \left(z^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}-1}\left((p-1)z^\alpha + \varepsilon\right)\widetilde{M}, \quad z \geqslant \widetilde{M}. \tag{1.5}$$

Due to the choice of α and ε (see (i)–(iii))

$$\begin{aligned} E'(\varepsilon) &= \frac{\widetilde{M}}{\alpha}\left(z^\alpha + \varepsilon\right)^{\frac{p-2-2\alpha}{\alpha}}\left[(p-2)\varepsilon + \left(p^2 - (3+\alpha)p + 2 + 2\alpha\right)z^\alpha\right] \\ &= \frac{\widetilde{M}}{\alpha}\left(z^\alpha + \varepsilon\right)^{\frac{p-2-2\alpha}{\alpha}}(p-2)\left[\varepsilon + (p-1-\alpha)z^\alpha\right] \geqslant 0 \quad \text{for } z \geqslant \widetilde{M}. \end{aligned}$$

Thus, taking into account that $E(\varepsilon) \geqslant E(0)$ and $h'(x) \geqslant \widetilde{M}$, from (1.4) we conclude that

$$Lh \geqslant (p-1)\widetilde{M}^{p-1}. \tag{1.6}$$

For the function

$$v(t,x) \equiv u(t,x) - h(x)$$

we have

$$Lu - Lh = u_t - a_\varepsilon(u_x)u_{xx} + a_\varepsilon(h_x)h_{xx} = v_t - a_\varepsilon(u_x)v_{xx} + \left(a_\varepsilon(h') - a_\varepsilon(u_x)\right)h''.$$

On the other hand due to (1.3) and (1.6) we obtain

$$Lu - Lh = \lambda g_M(u) - Lh \leqslant \lambda g_M(u) - (p-1)\widetilde{M}^{p-1}.$$

Hence

$$v_t - a_\varepsilon(u_x)v_{xx} \leqslant \left(a_\varepsilon(u_x) - a_\varepsilon(h')\right)h'' + \lambda g_M(u) - (p-1)\widetilde{M}^{p-1}.$$

Suppose that at a point $N \in \overline{Q}_T \setminus \Gamma_T$ the function $v(t,x)$ attains its positive maximum. Here Γ_T is the parabolic boundary of Q_T i.e. $\Gamma_T = \partial Q_T \setminus \{(t,x)\colon\ t = T,\ -l < x < l\}$. At the point N we have $v > 0$ and $v_x = 0$ or $u > h \geqslant 0$ and $u_x = h' \geqslant \widetilde{M}$ (in particular $a_\varepsilon(u_x) - a_\varepsilon(h') = 0$). Thus

$$v_t - a_\varepsilon(u_x)v_{xx}|_N \leqslant \lambda g_M(u) - (p-1)\widetilde{M}^{p-1}|_N \leqslant |\lambda| M^q - (p-1)l_*^{1-p}M^{p-1}.$$

Here we use the fact that for positive u we have $0 \leqslant g_M(u) \leqslant g(M) = M^q$. Hence due to (0.6)

$$v_t - a_\varepsilon(u_x)v_{xx}|_N < 0.$$

This contradicts the assumption that $v(t,x)$ attains positive maximum at N. Due to the homogeneous boundary conditions, for $x = \pm l$ we have $v = -h \leqslant 0$. Moreover

$$v(0,x) = u_0(x) - h(x) = u_0(x) - u_0(-l) - \left(h(x) - h(-l)\right) \leqslant K(x+l) - h'(\xi)(x+l) \leqslant 0,$$

$\xi \in [-l, x]$. Here we use the fact that $h' \geqslant \widetilde{M} \geqslant K$. Taking into account that $v(t,x)$ cannot attain positive maximum in $\overline{Q}_T \setminus \Gamma_T$ we conclude that

$$v(t,x) \leqslant 0 \quad \text{or} \quad u(t,x) \leqslant h(x) \quad \text{in } \overline{Q}_T.$$

Now let us obtain the estimate from the below. For the function $w(t,x) \equiv u(t,x) + h(x)$ we have

$$Lu + Lh = u_t - a_\varepsilon(u_x)u_{xx} - a_\varepsilon(h_x)h_{xx} = w_t - a_\varepsilon(u_x)w_{xx} - \big(a_\varepsilon(h') - a_\varepsilon(u_x)\big)h''.$$

On the other hand

$$Lu + Lh = \lambda g_M(u) + Lh \geqslant \lambda g_M(u) + (p-1)\widetilde{M}^{p-1}.$$

Thus

$$w_t - a_\varepsilon(u_x)w_{xx} \geqslant \big(a_\varepsilon(h') - a_\varepsilon(u_x)\big)h'' + \lambda g_M(u) + (p-1)\widetilde{M}^{p-1}.$$

Suppose that at a point $N_1 \in \overline{Q}_T \setminus \Gamma_T$ the function $w(t,x)$ attains its negative minimum. At this point we have $w < 0$ and $w_x = 0$ or $u < -h \leqslant 0$ and $u_x = -h' \leqslant -\widetilde{M}$. Therefore (because $a_\varepsilon(z) = a_\varepsilon(-z)$)

$$w_t - a_\varepsilon(u_x)w_{xx}|_{N_1} \geqslant \lambda g_M(u) + (p-1)\widetilde{M}^{p-1}|_{N_1} \geqslant -|\lambda|M^q + (p-1)l_*^{1-p}M^{p-1}. \tag{1.7}$$

Here we use the inequality

$$\lambda g_M\big(u(N_1)\big) \geqslant -|\lambda|g(M) = -|\lambda|M^q.$$

If $\lambda \geqslant 0$, then the last inequality follows from the fact that $g_M(u) \geqslant -g(M)$. If $\lambda < 0$, then the inequality follows from the fact that $g_M(u) \leqslant g(M)$. Hence due to (0.6) from (1.7) we obtain

$$w_t - a_\varepsilon(u_x)w_{xx}|_{N_1} > 0.$$

This contradicts the assumption that $w(t,x)$ attains negative minimum at N_1.

Due to the homogeneous boundary conditions, for $x = \pm l$ we have $w = h \geqslant 0$. Moreover,

$$w(0,x) = u_0(x) + h(x) = u_0(x) - u_0(-l) + h(x) - h(-l) \geqslant -K(x+l) + h'(\xi)(x+l) \geqslant 0.$$

Taking into account that $w(t,x)$ cannot attain negative minimum in $\overline{Q}_T \setminus \Gamma_T$ we conclude that

$$w(t,x) \geqslant 0 \quad \text{or} \quad u(t,x) \geqslant -h(x) \quad \text{in } \overline{Q}_T.$$

Finally we obtain

$$-h(x) \leqslant u(t,x) \leqslant h(x). \tag{1.8}$$

Now taking $\tilde{h}(x) \equiv h(-x)$ instead of $h(x)$ we obtain

$$-\tilde{h}(x) \leqslant u(t,x) \leqslant \tilde{h}(x). \tag{1.9}$$

Estimate (1.9) can be easily established in the same way as (1.8) due to the fact that $\tilde{h}'^{\alpha}(x) \geqslant \widetilde{M}^{\alpha}$ and $\tilde{h}''(x) = -\widetilde{M}$. The first inequality ($\tilde{h}'^{\alpha}(x) \geqslant \widetilde{M}^{\alpha}$) follows from $-\tilde{h}'(x) \geqslant \widetilde{M} \geqslant 0$ due to the choice of α.

From (1.8) and (1.9) we conclude that

$$\big|u(t,x)\big| \leqslant h(0) = \tilde{h}(0) = l_*\widetilde{M} = M.$$

Lemma 1 is proved. □

Remark 2. Actually Lemma 1 gives us not only the estimate of $\max|u|$ but also the boundary gradient estimate. In fact, from (1.8) it follows that

$$\big|u_x(t,-l)\big| \leqslant h'(-l) = M\frac{2+4l}{3l^2+2l}.$$

Similarly, from (1.9) we obtain

$$\big|u_x(t,l)\big| \leqslant -\tilde{h}'(l) = M\frac{2+4l}{3l^2+2l}.$$

Let us turn to the global gradient estimate. We will use here the classical Kruzhkov's idea of introducing of a new spatial variable (see, for example, [7]). Define the function $H(\tau)$ by the following:

$$H(\tau) = -C\frac{\tau^2}{2} + \left[C(1+2l)+\epsilon\right]\tau, \quad \tau \in [0, 2l],$$

where

$$C = \max\left\{K, \frac{4l+2}{3l^2+2l}M, \left(\frac{|\lambda|M^q}{p-1}\right)^{\frac{1}{p-1}}\right\}.$$

Obviously

$$H'' = -C, \qquad H' \geqslant C + \epsilon > C.$$

Lemma 2. *If conditions* (0.4)–(0.6) *are fulfilled, then for any classical solution of problem* (1.1), (0.2), (0.3) *the following estimate is valid*:

$$\left|u_x(t,x)\right| \leqslant (1+2l)C.$$

Proof. Consider Eq. (1.1) at two different points (t,x) and (t,y) $(x \neq y)$. Taking into account the fact that due to Lemma 1 $g(u) = g_M(u)$, we have

$$u_t(t,x) - a_\varepsilon\big(u_x(t,x)\big)u_{xx}(t,x) = \lambda g\big(u(t,x)\big), \tag{1.10}$$
$$u_t(t,y) - a_\varepsilon\big(u_y(t,y)\big)u_{yy}(t,y) = \lambda g\big(u(t,y)\big). \tag{1.11}$$

Subtracting Eq. (1.11) from (1.10) for

$$\mathrm{v}(t,x,y) \equiv u(t,x) - u(t,y)$$

we obtain

$$\mathrm{v}_t - a_\varepsilon\big(u_x(t,x)\big)\mathrm{v}_{xx} - a_\varepsilon\big(u_y(t,y)\big)\mathrm{v}_{yy} = \lambda\Big(g\big(u(t,x)\big) - g\big(u(t,y)\big)\Big). \tag{1.12}$$

Consider (1.12) in the domain

$$P = \big\{(t,x,y)\colon\ t \in (0,T),\ x \in (-l,l),\ y \in (-l,l),\ x > y\big\}.$$

For

$$\mathrm{w}(t,x,y) = \mathrm{v}(t,x,y) - H(x-y)$$

we have

$$\begin{aligned}\mathrm{w}_t - a_\varepsilon\big(u_x(t,x)\big)\mathrm{w}_{xx} - a_\varepsilon\big(u_y(t,y)\big)\mathrm{w}_{yy} &= \Big(a_\varepsilon\big(u_x(t,x)\big) + a_\varepsilon\big(u_y(t,y)\big)\Big)H'' + \lambda\Big(g\big(u(t,x)\big) - g\big(u(t,y)\big)\Big)\\ &\leqslant -C\Big(a_\varepsilon\big(u_x(t,x)\big) + a_\varepsilon\big(u_y(t,y)\big)\Big) + 2|\lambda|M^q.\end{aligned} \tag{1.13}$$

Suppose that at a point $N \in \overline{P} \setminus \Gamma$ the function $\mathrm{w}(t,x,y)$ attains its maximum. At this point we have $\mathrm{w}_x = \mathrm{w}_y = 0$, or

$$u_x = u_y = H' > C \geqslant (2l+1)\widetilde{M} > \widetilde{M}.$$

Hence from (1.13) we have (recall that a_ε is nondecreasing with respect to ε for $z \geqslant \widetilde{M}$)

$$\mathrm{w}_t - a_\varepsilon\big(u_x(t,x)\big)\mathrm{w}_{xx} - a_\varepsilon\big(u_y(t,y)\big)\mathrm{w}_{yy}|_N < -2(p-1)C^{p-1} + 2|\lambda|M^q \leqslant 0$$

due to the choice of C. This contradicts the assumption that $\mathrm{w}(t,x,y)$ attains maximum at the internal point of the domain P.

Now consider $\mathrm{w}(t,x,y)$ on Γ. The parabolic boundary of P consists of four parts:

(1) $x = y$,
(2) $y = -l$, $x \in [-l, l]$,

(3) $x = l$, $y \in [-l, l]$, and
(4) $t = 0$.

On the first part we obviously have $\mathrm{w} = -H(0) = 0$. On the second and the third parts we have, respectively,

$$\mathrm{w} = u(t,x) - H(x+l) \leqslant 0 \quad \text{and} \quad \mathrm{w} = -u(t,y) - H(l-y) \leqslant 0.$$

The first inequality follows from (1.8), the fact that $h(-l) = H(0) = 0$ and

$$H' \geqslant C \geqslant \frac{4l+2}{3l^2+2l} M = (1+2l)\widetilde{M} \geqslant h'.$$

Concerning the second one note that due to (1.9) we have to prove that $\tilde{h}(y) \leqslant H(l-y)$. Put $l - y = \mu$, $\mu \in [0, 2l]$. Thus one has to prove now that $\tilde{h}(l-\mu) \leqslant H(\mu)$. Obviously $\tilde{h}(l) = H(0) = 0$. Moreover one can easily see that

$$\tilde{h}_\mu = -\tilde{h}'(l-\mu) \leqslant \widetilde{M}(1+2l) \leqslant C \leqslant H'.$$

That means that $\tilde{h}(l-\mu) \leqslant H(\mu)$ for $\mu \in [0, 2l]$ or $\tilde{h}(y) \leqslant H(l-y)$ for $y \in [-l, l]$.

For $t = 0$ we have

$$u_0(x) - u_0(y) - H(x-y) \leqslant K(x-y) - \big(H(x-y) - H(0)\big) \leqslant K(x-y) - C(x-y) \leqslant 0.$$

Consequently $\mathrm{w}(t,x,y) \leqslant 0$ in $\overline{P}$, which means

$$u(t,x) - u(t,y) \leqslant H(x-y) \quad \text{in } \overline{P}.$$

Similarly, taking the function $\tilde{\mathrm{v}} \equiv u(t,y) - u(t,x)$ instead of v, we obtain that

$$u(t,y) - u(t,x) \leqslant H(x-y) \quad \text{in } \overline{P}$$

and as a consequence we conclude that

$$\big|u(t,x) - u(t,y)\big| \leqslant H(x-y) \quad \text{in } \overline{P}.$$

Using the symmetry of the variables x and y, we consider the case $y > x$ in the same way. As a result we obtain that for $x \in [-l, l]$, $y \in [-l, l]$, $|x-y| > 0$ the following inequality holds:

$$\frac{|u(t,x) - u(t,y)|}{|x-y|} \leqslant \frac{H(|x-y|) - H(0)}{|x-y|},$$

which in turn implies the estimate

$$\big|u_x(t,x)\big| \leqslant H'(0) = (1+2l)C + \epsilon.$$

Passing to the limit when $\epsilon \to 0$ we conclude

$$\big|u_x(t,x)\big| \leqslant (1+2l)C.$$

Lemma 2 is proved. □

Lemma 3. *If $p \geqslant 3$, then for any classical solution of problem* (1.1), (0.2), (0.3) *the following estimate holds*:

$$\|u_t\|^2_{L_2(Q_T)} \leqslant 2lT\big(|\lambda|M^q\big)^2 + \frac{2}{p}\int_{-l}^{l}\big(u_{0x}^2 + 1\big)^{\frac{p}{2}}\,dx.$$

Here, without loss of generality, we assume that $\varepsilon \leqslant 1$.

Proof. Recall that for $p \geqslant 3$ we take $\alpha = 2$. Multiply (1.1) by u_t, taking into account that $g_M(u) = g(u)$ and integrating by parts with respect to x we obtain

$$\int_{-l}^{l} \left(u_t^2 + \left(u_x^2 + \varepsilon\right)^{\frac{p-2}{2}} u_x u_{tx}\right) dx = \int_{-l}^{l} \lambda g(u) u_t \, dx.$$

Use the Young inequality to obtain

$$\int_{-l}^{l} u_t^2 \, dx + \frac{1}{p} \frac{d}{dt} \int_{-l}^{l} \left(u_x^2 + \varepsilon\right)^{\frac{p}{2}} dx \leqslant \frac{1}{2} \int_{-l}^{l} u_t^2 \, dx + \frac{1}{2} \int_{-l}^{l} \left(\lambda g(u)\right)^2 dx.$$

Integrating with respect to t we conclude that

$$\int_{Q_T} u_t^2 \, dx \, dt \leqslant 2lT\left(|\lambda| M^q\right)^2 + \frac{2}{p} \int_{-l}^{l} \left(u_{0x}^2 + 1\right)^{\frac{p}{2}} dx.$$

Lemma 3 is proved. □

2. Existence and uniqueness

Consider equation

$$u_{\varepsilon t} - \left(\left(u_{\varepsilon x}^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}} u_{\varepsilon x}\right)_x = \lambda g(u_\varepsilon). \tag{2.1}$$

The classical solvability of problem (2.1), (0.2), (0.3) for $\varepsilon > 0$ follows from Lemmas 1, 2 (see, for example, [2]). Multiply (2.1) by $\phi \in L_2(0, T; \mathring{H}^1(-l, l))$ and integrate by parts (with respect to x) to obtain

$$\int_{Q_T} \left(u_{\varepsilon t}\phi + \left(u_{\varepsilon x}^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}} u_{\varepsilon x}\phi_x\right) dx \, dt = \int_{Q_T} \lambda g(u_\varepsilon)\phi \, dx \, dt. \tag{2.2}$$

From the estimates of the previous section it follows that the right-hand side of (2.1) is uniformly bounded independently of ε. Hence from [5] (see [5, Theorem 3.1]) it follows that for $v_\varepsilon = u_{\varepsilon x}$ we have

$$\left|v_\varepsilon(t, x) - v_\varepsilon(\tau, y)\right| \leqslant C_1\left(|x - y| + |t - \tau|\right)^\beta, \quad \forall (t, x), (\tau, y) \in Q_T,$$

where constants $C_1, \beta \in (0, 1)$ are independent of ε. Moreover, from (2.1) it follows that

$$\|u_{\varepsilon t}\|_{L_2(0,T;H^{-1}(-l,l))} \leqslant C_2$$

with constant C_2 independent of ε. Rewrite (2.2) in the following form:

$$\int_0^T \langle u_{\varepsilon t}, \phi\rangle \, dt + \int_{Q_T} \left(u_{\varepsilon x}^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}} u_{\varepsilon x}\phi_x \, dx \, dt = \int_{Q_T} \lambda g(u_\varepsilon)\phi \, dx \, dt. \tag{2.3}$$

Based on the above estimates we conclude that there exists subsequence ε_n such that

$$u_{\varepsilon_n} \to u, \quad \frac{\partial u_{\varepsilon_n}}{\partial x} \to \frac{\partial u}{\partial x} \quad \text{uniformly in } C^0(Q_T),$$

$$\left(\left(\frac{\partial u_{\varepsilon_n}}{\partial x}\right)^\alpha + \varepsilon\right)^{\frac{p-2}{\alpha}} \to \left|\frac{\partial u}{\partial x}\right|^{p-2} \quad \text{uniformly in } C^0(Q_T),$$

$$\left|\frac{\partial u_{\varepsilon_n}}{\partial x}\right|^{r-1} \frac{\partial u_{\varepsilon_n}}{\partial x} \to \left|\frac{\partial u}{\partial x}\right|^{r-1} \frac{\partial u}{\partial x} \quad \text{uniformly in } C^0(Q_T),$$

and

$$\frac{\partial u_{\varepsilon_n}}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L_2\big(0,T;H^{-1}(-l,l)\big).$$

Passing to the limit in (2.3) we obtain the required solution.

Remark 3. Let us mention here that due to Lemma 3, if $p \geqslant 3$, then we have

$$\frac{\partial u_{\varepsilon_n}}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } L_2(Q_T)$$

and we can pass to the limit in (2.2). Thus the obtained global generalized solution for $p \geqslant 3$ is somehow better with respect to t. Namely $u_t \in L_2(Q_T)$. For that case in the definition of the global generalized solution instead of

$$\int_0^T \langle u_t, \phi\rangle\, dt \quad \text{we take} \int_{Q_T} u_t \phi \, dx\, dt.$$

Let us prove the uniqueness. Suppose that there exist two solutions u_1 and u_2. For $u = u_1 - u_2$ we have

$$\int_0^T \langle u_t, \phi\rangle\, dt + \int_{Q_T} \big(|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x}\big)\phi_x\, dx\, dt = \int_{Q_T} \big(\lambda\big(g(u_1) - g(u_2)\big)\big)\phi\, dx\, dt.$$

By setting u instead of ϕ we obtain

$$\int_0^T \langle u_t, u\rangle\, dt + \int_{Q_T} \big(|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x}\big)(u_{1x} - u_{2x})\, dx\, dt \leqslant \int_{Q_T} \lambda G u^2\, dx\, dt$$

due to the Lipschitz continuity of g,

$$\big|g(u_1) - g(u_2)\big| \leqslant G|u_1 - u_2| = G|u|.$$

Hence

$$\int_0^T \langle u_t, u\rangle\, dt \leqslant \int_{Q_T} \lambda G u^2\, dx\, dt \tag{2.4}$$

since

$$\int_{Q_T} \big(|u_{1x}|^{p-2}u_{1x} - |u_{2x}|^{p-2}u_{2x}\big)(u_{1x} - u_{2x})\, dx\, dt \geqslant 0.$$

The latter is due to the monotonicity of operator $A(u) : u \in \mathring{H}^1 \to A(u) \in H^{-1}$ defined by

$$\big\langle A(u), w\big\rangle = \int_{-l}^{l} |u_x|^{p-2} u_x w_x\, dx$$

(for more details see [6]).

Notice that instead of integrating from 0 to T we can integrate from 0 to t for any $t \in (0,T]$ hence from (2.4) we conclude that

$$\|u\|^2_{L_2(-l,l)} \leqslant \int_0^t 2|\lambda| G \|u\|^2_{L_2(-l,l)}\, d\tau.$$

Here we use the fact that

$$\frac{d}{dt}\|u\|^2_{L_2(-l,l)} = 2\langle u_t, u\rangle.$$

From Gronwall's inequality we conclude that $u_1 \equiv u_2$. The theorem is proved.

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