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On the global solvability of the Cauchy problem for a quasilinear ultraparabolic equation

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Dedicated to the memory of my father Professor S.A. Tersenov

Abstract. In the present paper we study the Cauchy problem as well as the first initial boundary value problem for a class of quasilinear ultraparabolic equations. We show that the presence of the low order term satisfying a certain assumption provides a global solvability of the above problems. The optimality of this assumption is demonstrated.

Keywords: partial diffusivity, a priori estimates, global solvability

1. Introduction and formulation of the results

The present work deals with the solvability of the following ultraparabolic equation

$$u_t + g(t,u)u_x + f(t,u) = \Delta_{\mathbf{y}} u \quad \text{in } S_T = (0,T) \times \mathbf{R} \times \mathbf{R}^{\mathbf{n}}, \tag{1.1}$$

coupled with initial condition

$$u(0,x,\mathbf{y}) = u_0(x,\mathbf{y}), \tag{1.2}$$

here $\Delta_{\mathbf{y}} = \sum_1^n \partial^2/\partial y_i^2$.

Equation (1.1) describes nonstationary transport (of matter, impulse, temperature) processes where in some direction the effect of the diffusion is negligible as compared to the convection (see [12,15,17]). Such equations appear in age dependent population diffusion (see [5]) and in mathematical finance (see [3,9,11]) as well. This class of equations has received considerable attention in the recent decades from different authors (see, e.g., [1,2,4,6,10,13,14,21–23] and the references therein).

The local existence of a smooth solution (at least Lipschitz continuous with respect to $x, \mathbf{y}$ and Hölder continuous with respect to t) to problem (1.1), (1.2) was proved in [14] under the assumptions that g,

f and u_0 are globally Lipschitz continuous functions, the global classical solvability was obtained if additionally u_0 is nonincreasing with respect to x and $g(u)-g(v) \geqslant c_0(u-v)$ for some positive constant c_0. Our goal is to prove the global solvability of problem (1.1), (1.2) for locally Hölder continuous functions g and f under the following structure restriction:

$$K\big|g(t,u_2)-g(t,u_1)\big| \leqslant f(t,u_2)-f(t,u_1) \quad \text{for } u_1<u_2, t\in[0,T), \tag{1.3}$$

where $T>0$ is an arbitrary positive constant and

$$K=\sup_{\mathbf{R}\times\mathbf{R}^n}\frac{|u_0(x,\mathbf{y})-u_0(x',\mathbf{y})|}{|x-x'|}. \tag{1.4}$$

Actually we show that the presence of low order term f connected with g as in (1.3) provides global solvability and, as it will be shown below (Example 4), condition (1.3) is in some sense optimal.

Definition 1. We say that function $u(t,x,\mathbf{y})$ is a generalized solution of problem (1.1), (1.2) if:

(i) $u\in L^\infty(S_T)\cap L^2(S_T)\cap C_t^{1/2}(S_T)$, $u_x\in L^\infty(S_T)$, $u_{y_i}\in L^\infty(S_T)\cap L^2(S_T)$, $u_t, u_{y_iy_j}\in L^2_{\text{loc}}(S_T)$, $i,j=1,\dots,n$.

(ii) u satisfies Eq. (1.1) almost everywhere in S_T, initial condition is admitted in the classical sense.

Here $C_t^{1/2}$ is the set of Hölder continuous with respect to t functions with Hölder exponent $1/2$.

Theorem 1. *Suppose that u_0 is a Lipschitz continuous function vanishing at infinity (when $|x|+|\mathbf{y}|\to\infty$) so that $\|u_0\|_{L^2(\mathbf{R}^{n+1})}<\infty$. Assume that g, f are locally Hölder continuous functions such that $g(t,0)=f(t,0)=0$. If condition (1.3) is fulfilled, then for arbitrary $T>0$ there exists a generalized solution of problem (1.1), (1.2) and for this solution the following estimates take place:*

$$\|u\|_{L^\infty(S_T)}\leqslant M=\sup_{\mathbf{R}\times\mathbf{R}^n}\big|u_0(x,\mathbf{y})\big|,$$

$$\|u_x\|_{L^\infty(S_T)}\leqslant K=\sup_{\mathbf{R}\times\mathbf{R}^n}\frac{|u_0(x,\mathbf{y})-u_0(x',\mathbf{y})|}{|x-x'|},$$

$$\|u_{y_i}\|_{L^\infty(S_T)}\leqslant K_i=\sup_{\mathbf{R}\times\mathbf{R}^n}\frac{|u_0(x,\mathbf{y})-u_0(x,\mathbf{y}')|}{|y_i-y_i'|},\quad i=1,\dots,n.$$

If, in addition, function $g(t,u)$ is differentiable with respect to u, then this solution is unique.

Here $\mathbf{y}'=(y_1,\dots,y_{i-1},y_i',y_{i+1},\dots,y_n)$ (for $i=1$ and $i=n$ we put $\mathbf{y}'=(y_1',y_2,\dots,y_n)$ and $\mathbf{y}'=(y_1,\dots,y_{n-1},y_n')$, respectively).

Let us give three examples when condition (1.3) is fulfilled for an arbitrary $T>0$.

Example 1. Consider the equation

$$u_t+g(t,u)u_x+\lambda g(t,u)=\Delta_{\mathbf{y}}u,$$

where $g(t,0)=0$. Condition (1.3) is fulfilled for arbitrary $T\in(0,+\infty)$ if locally Hölder continuous function g is nondecreasing with respect to u and the constant $\lambda\geqslant K$.

Example 2. For the equation

$$u_t + au^{2/3}u_x + \lambda u^{1/3} = \Delta_{\mathbf{y}} u,$$

where $a, \lambda \in \mathbf{R}$, condition (1.3) takes the form $K|a||u_2^{1/3} + u_1^{1/3}| \leqslant \lambda$. Taking into account that here $m \equiv \max|u| = \max|u_0|$ we conclude that (1.3) is fulfilled for an arbitrary $T \in (0, +\infty)$ if

$$\lambda \geqslant 2m^{1/3}|a|K.$$

Example 3. For the equation

$$u_t + au^2 u_x + \lambda u = \Delta_{\mathbf{y}} u,$$

condition (1.3) is fulfilled for arbitrary $T > 0$ if

$$\lambda \geqslant 2m|a|K, \quad m = \max|u_0|.$$

In the next example we will demonstrate the optimality of condition (1.3).

Example 4. Consider equation

$$u_t + auu_x + \lambda u = \Delta_{\mathbf{y}} u \tag{1.5}$$

and suppose that initial data does not depend on $\mathbf{y}$

$$u(0, x, \mathbf{y}) = u_0(x). \tag{1.6}$$

Condition (1.3) takes the form $\lambda \geqslant |a|K$, i.e. if this inequality holds then there exists a solution of problem (1.5), (1.6) for $T \in (0, +\infty)$. The solution of the equation

$$u_t + auu_x + \lambda u = 0 \tag{1.7}$$

coupled with condition (1.6) is at the same time solution of (1.5), (1.6).

It is well known that, even for smooth initial data, the solution of problem (1.7), (1.6) can develop shocks in finite time. Global Lipschitz continuous solution of this problem exists only if characteristics do not intersect. The family of characteristics $x = x(t)$ is defined by

$$x(t) = x(0) + \frac{a}{\lambda}\phi\bigl(x(0)\bigr)\bigl(1 - \mathrm{e}^{-\lambda t}\bigr).$$

Suppose that the characteristics $x_1(t)$ and $x_2(t)$ which start from the points $x_1(0)$ and $x_2(0)$, respectively, intersect at time $t^* \in (0, +\infty)$, i.e. $x_1(t^*) = x_2(t^*)$ or

$$x_1(0) + \frac{a}{\lambda}\phi\bigl(x_1(0)\bigr)\bigl(1 - \mathrm{e}^{-\lambda t^*}\bigr) = x_2(0) + \frac{a}{\lambda}\phi\bigl(x_2(0)\bigr)\bigl(1 - \mathrm{e}^{-\lambda t^*}\bigr)$$

or

$$\frac{|\phi(x_1(0)) - \phi(x_2(0))|}{|x_1(0) - x_2(0)|} = \frac{\lambda}{|a|}\frac{\mathrm{e}^{\lambda t^*}}{\mathrm{e}^{\lambda t^*} - 1}.$$

The last is possible only if $\lambda < |a|K$, hence condition $\lambda \geqslant |a|\,K$ is optimal.

Let us mention here the example in [2] where it was shown that a smooth solution of equation $u_t - uu_x = u_{yy}$ starting from smooth and compactly supported initial data depending on both x and y becomes discontinuous after a finite time.

Obviously if condition (1.3) is fulfilled only for some value of T (say T^*), then we have the local existence (i.e., we guarantee the existence on the interval $[0, T^*)$). What happens if condition (1.3) is not fulfilled? In this case we also guarantee the local solvability of the problem under an additional assumptions.

Theorem 2. *Suppose that u_0 is a Lipschitz continuous function vanishing at infinity (when $|x| + |\mathbf{y}| \to \infty$) so that $\|u_0\|_{L^2(\mathbf{R}^{n+1})} < \infty$. Assume that functions $g(t,u)$ and $f(t,u)$ are locally Hölder continuous with respect to t and locally Lipschitz continuous with respect to u such that $g(t,0) = f(t,0) = 0$, $uf(t,u) \geqslant 0$. Then for some $T^* > 0$ there exists a generalized solution of problem (1.1), (1.2). If g and f are globally Lipschitz continuous functions with respect to u, then condition $uf(t,u) \geqslant 0$ is unnecessary.*

This solution is unique if in addition function $g(t,u)$ is differentiable with respect to u.

Note that the estimate of T^* from the below can be find in the explicit form (see Examples 5, 6 and the proof of Theorem 2 in Section 3).

Let us turn now to the initial boundary value problem.

$$u_t + g(t,u)u_x + f(t,u) = \Delta_{\mathbf{y}} u \quad \text{in } S_T^l = (0,T) \times Q^l, \tag{1.8}$$

where $Q^l = (-l,l) \times (-l_1,l_1) \times \cdots \times (-l_n,l_n)$, coupled with initial condition

$$u(0,x,\mathbf{y}) = u_0(x,\mathbf{y}) \quad \text{in } Q^l, \quad u_0|_{x=\pm l} = u_0|_{y_i=\pm l_i} = 0 \tag{1.9}$$

and boundary conditions

$$u|_{x=\pm l} = u|_{y_i=\pm l_i} = 0, \quad i = 1,\ldots,n. \tag{1.10}$$

Note that due to the specificity of the equation (namely $g(t,0) = f(t,0) = 0$) even though only the first-order derivative with respect to x is present in (1.8) we have two boundary conditions in x direction. If $g > 0$ ($g < 0$) than we should not impose condition on $x = l$ ($x = -l$). See also comments in Section 3.

Definition 2. We say that function $u(t,x,\mathbf{y})$ is a generalized solution of problem (1.8)–(1.10) if:

(i) $u \in C_t^{1/2}(S_T^l)$, $u_x, u_{y_i} \in L^\infty(S_T^l)$, $u_t, u_{y_iy_j} \in L^2(S_T^l)$, $i,j = 1,\ldots,n$.

(ii) u satisfies Eq. (1.8) almost everywhere in S_T^l, initial and boundary conditions are admitted in the classical sense;

For simplicity we use the same symbols (M, K, K_i) as in Theorem 1.

Theorem 3. *Suppose that u_0 is a Lipschitz continuous function and g, f are locally Hölder continuous functions such that $g(t,0) = f(t,0) = 0$. If condition* (1.3) *is fulfilled, then there exists a unique generalized solution of problem* (1.8)–(1.10) *and for this solution the following estimates take place*:

$$\|u\|_{L^\infty(S_T^l)} \leqslant M = \sup_{Q^l} \bigl|u_0(x,\mathbf{y})\bigr|,$$

$$\|u_x\|_{L^\infty(S_T^l)} \leqslant K = \sup_{Q^l} \frac{|u_0(x,\mathbf{y}) - u_0(x',\mathbf{y})|}{|x - x'|},$$

$$\|u_{y_i}\|_{L^\infty(S_T^l)} \leqslant K_i = \sup_{Q^l} \frac{|u_0(x,\mathbf{y}) - u_0(x,\mathbf{y}')|}{|y_i - y_i'|}, \quad i = 1,\ldots,n.$$

If, in addition, function $g(t,u)$ is differentiable with respect to u, then this solution is unique.

Similarly to the Cauchy problem the following theorem takes place.

Theorem 4. *Suppose that u_0 is a Lipschitz continuous function and functions $g(t,u)$ and $f(t,u)$ are locally Hölder continuous with respect to t and locally Lipschitz continuous with respect to u such that $g(t,0) = f(t,0) = 0$, $uf(t,u) \geqslant 0$. Then for some $T^* > 0$ there exists a generalized solution of problem* (1.8)–(1.10). *If g and f are globally Lipschitz continuous functions with respect to u, then condition $uf(t,u) \geqslant 0$ is unnecessary.*

This solution is unique if in addition function $g(t,u)$ is differentiable with respect to u.

Let us mention that Examples 1–4 can be trivially extended to the initial – boundary value problem.

In the next section we obtain the a priori estimate for the regularized problem and in the last section we prove Theorems 1–4.

2. A priori estimates for the auxiliary problem

Approximate problem (1.8)–(1.10) by the following one

$$u_t^\varepsilon + g\bigl(t,u^\varepsilon\bigr)u_x^\varepsilon + f\bigl(t,u^\varepsilon\bigr) = \Delta_{\mathbf{y}} u^\varepsilon + \varepsilon u_{xx}^\varepsilon \quad \text{in } S_T^l = (0,T) \times Q^l, \tag{2.1}$$

$$u^\varepsilon(0,x,\mathbf{y}) = u_0(x,\mathbf{y}), \tag{2.2}$$

$$u^\varepsilon(t,x,\mathbf{y})|_{x=\pm l} = u^\varepsilon(t,x,\mathbf{y})|_{y_i=\pm l_i} = 0, \quad i = 1,\ldots,n. \tag{2.3}$$

The goal of the present section is to obtain a priori estimate of the solution of problem (2.1)–(2.3) independent of ε, l and l_i. The key step is the estimate of the derivative u_x^ε. We apply the approach used in [18,20] and which is a development of the Kruzhkov's idea of introducing of a new spatial variable [7].

To simplify the notation, below we will omit the superscript ε.

We say that u is a classical solution of (2.1)–(2.3) if $u \in C^{1;2}_{t;x,\mathbf{y}}(S^l_T) \cap C(\bar{S}^l_T)$. Here $C^{1;2}_{t;x,\mathbf{y}}(S^l_T)$ is the set of functions having the first derivative with respect to t and the second derivatives with respect to $x, \mathbf{y}$ continuous in S^l_T.

Everywhere below by Γ we denote the parabolic boundary of S^l_T, i.e.

$$\Gamma \equiv \partial S^l_T \setminus \bigl\{t = T, |x| < l, |y_i| < l_i, i = 1, \dots, n\bigr\}$$

and by L_0 we denote the operator

$$L_0 \equiv \frac{\partial}{\partial t} - \varepsilon \frac{\partial^2}{\partial x^2} - \Delta_{\mathbf{y}}.$$

To simplify the calculations we substitute condition (1.3) by the following one

$$K\bigl|g(t, u_2) - g(t, u_1)\bigr| < f(t, u_2) - f(t, u_1) \quad \text{for } u_1 < u_2, t \in [0, T) \tag{2.4}$$

and then explain how we extend the results to the case of nonstrict inequality (1.3).

Lemma 2.1. *Let $u(t, x, \mathbf{y})$ be a classical solution of problem* (2.1)–(2.3). *Assume that condition* (2.4) *is fulfilled*, *then*

$$\bigl|u(t, x, \mathbf{y})\bigr| \leqslant M.$$

Proof. From condition (2.4) and the assumption $f(t, 0) = g(t, 0) = 0$ it follows that $u\, f(t, u) > 0$ for $u \neq 0$ which implies that u cannot attain neither its positive maximum nor its negative minimum in $\bar{S}^l_T \setminus \Gamma$. Taking into account boundary conditions (2.3) we obtain the needed estimate. □

Lemma 2.2. *Let $u(t, x, \mathbf{y})$ be a classical solution of problem* (2.1)–(2.3) *and assume that condition* (2.4) *is fulfilled*. *Then*

$$\bigl|u(t, x, \mathbf{y})\bigr| \leqslant K(l + x) \quad \textit{and} \quad \bigl|u(t, x, \mathbf{y})\bigr| \leqslant K(l - x) \quad \textit{in } \bar{S}^l_T.$$

Proof. For the function $u(t, x, \mathbf{y}) - K(l + x)$ we have

$$L_0\bigl(u - K(l + x)\bigr) = -g(t, u)u_x - f(t, u).$$

If $u - K(l + x)$ attains its positive maximum at the point $N \in \bar{S}^l_T \setminus \Gamma$ then at this point

$$u > 0 \quad \text{and} \quad u_x = K,$$

hence, due to (2.4),

$$L_0\bigl(u - K(l + x)\bigr)\big|_N = -g(t, u)K - f(t, u)|_N < 0.$$

This contradicts the assumption that $u - K(l + x)$ attains positive maximum at N. It is clear that

$$u|_\Gamma \leqslant 0$$

and hence

$$u(t,x,\mathbf{y}) - K(l+x) \leqslant 0 \quad \text{in } \bar{S}^l_T.$$

Now consider the function $u(t,x,\mathbf{y}) + K(l+x)$. If $u + K(l+x)$ attains its negative minimum at the point $N_1 \in \bar{S}^l_T \setminus \Gamma$ then at this point we have

$$u < 0, \qquad u_x = -K,$$

and hence, due to (2.4),

$$L_0\bigl(u + K(l+x)\bigr)\big|_{N_1} = g(t,u)K - f(t,u)|_{N_1} > 0.$$

This contradicts the assumption that $u + K(l+x)$ attains negative minimum at N_1. Taking into account that

$$u|_\Gamma \geqslant 0$$

we conclude that

$$u(t,x,\mathbf{y}) + K(l+x) \geqslant 0 \quad \text{in } \bar{S}^l_T.$$

The first inequality is proved, the second one can be proved similarly taking into account that (2.4) implies that $g(t,u)K - f(t,u) < 0$ for $u > 0$ and $-g(t,u)K - f(t,u) < 0$ for $u < 0$. $\square$

Lemma 2.3. *Let $u(t,x,\mathbf{y})$ be a classical solution of problem* (2.1)–(2.3) *and assume that condition* (2.4) *is fulfilled, then in $\bar{S}^l_T$ the following inequalities hold*

$$\bigl|u(t,x,\mathbf{y})\bigr| \leqslant K_i(l_i + y_i) \quad \textit{and} \quad \bigl|u(t,x,\mathbf{y})\bigr| \leqslant K_i(l_i - y_i), \quad i = 1,\dots,n.$$

Proof. Suppose that $u - K_i(l_i + y_i)$ attains its positive maximum at the point $N \in \bar{S}^l_T \setminus \Gamma$ then at this point $u > 0$ and $u_x = 0$, hence, taking into account that (2.4) implies the inequality $uf(t,u) > 0$ for $u \neq 0$, we conclude that

$$L_0\bigl(u - K_i(l_i + y_i)\bigr)\big|_N = -f(t,u)|_N < 0.$$

This contradicts the assumption that $u - K_i(l_i + y_i)$ attains positive maximum at N. Hence, taking into account that $u|_\Gamma \leqslant 0$, we obtain

$$u(t,x,\mathbf{y}) \leqslant K_i(l_i + y_i) \quad \text{in } \bar{S}^l_T.$$

If $u + K_i(l_i + y_i)$ attains its negative minimum at the point $N_i \in \bar{S}^l_T \setminus \Gamma$ then at this point we have $u < 0$ and $u_x = 0$, and hence, due to the fact that $uf(t,u) > 0$ for $u \neq 0$,

$$L_0\bigl(u + K_i(l_i + y_i)\bigr)\big|_{N_i} = -f(t,u)|_{N_i} > 0.$$

This contradicts the assumption that $u + K_i(l_i + y_i)$ attains negative minimum in the interior of the domain S^l_T. Consequently, taking into account that $u|_\Gamma \geqslant 0$, we conclude that

$$u(t, x, \mathbf{y}) \geqslant -K_i(l_i + y_i) \quad \text{in } \bar{S}^l_T.$$

Similarly we obtain the estimate $|u(t, x, \mathbf{y})| \leqslant K_i(l_i - y_i)$. $\square$

In the next two lemmas we will obtain the global estimate of u_x and $\nabla_{\mathbf{y}} u$.

Lemma 2.4. *Let* $u(t, x, \mathbf{y})$ *be a classical solution of problem* (2.1)–(2.3) *and assume that condition* (2.4) *is fulfilled. Then*

$$\bigl|u_x(t, x, \mathbf{y})\bigr| \leqslant K \quad \textit{in } \bar{S}^l_T.$$

Proof. Consider Eq. (2.1) in two different points $(t, x, \mathbf{y})$ and $(t, \xi, \mathbf{y})$:

$$u_t + g(t, u)u_x + f(t, u) = \varepsilon u_{xx} + \Delta_{\mathbf{y}} u, \quad u = u(t, x, \mathbf{y}), \tag{2.5}$$

$$u_t + g(t, u)u_\xi + f(t, u) = \varepsilon u_{\xi\xi} + \Delta_{\mathbf{y}} u, \quad u = u(t, \xi, \mathbf{y}). \tag{2.6}$$

Subtracting Eq. (2.6) from (2.5) for the function

$$w(t, x, \xi, \mathbf{y}) \equiv u(t, x, \mathbf{y}) - u(t, \xi, \mathbf{y}) - K(x - \xi)$$

we obtain

$$\begin{aligned} w_t - \varepsilon w_{xx} - \varepsilon w_{\xi\xi} - \Delta_{\mathbf{y}} w = {} & -\bigl(g\bigl(t, u(t, x, \mathbf{y})\bigr)u_x(t, x, \mathbf{y}) - g\bigl(t, u(t, \xi, \mathbf{y})\bigr)u_\xi(t, \xi, \mathbf{y})\bigr) \\ & - \bigl(f\bigl(t, u(t, x, \mathbf{y})\bigr) - f\bigl(t, u(t, \xi, \mathbf{y})\bigr)\bigr). \end{aligned}$$

Consider this equation in the domain

$$P = \bigl\{(t, x, \xi, \mathbf{y})\colon\ 0 < t < T, -l < \xi < x < l, |y_i| < l_i, i = 1, \ldots, n\bigr\}.$$

Denote by Γ_P the parabolic boundary of P. Suppose that function w attains its positive maximum at some point $N \in P \setminus \Gamma_P$, then at this point we have

$$w > 0, \qquad w_x = w_\xi = 0 \quad \text{i.e. } u(t, x, \mathbf{y}) > u(t, \xi, \mathbf{y}), u_x(t, x, \mathbf{y}) = u_\xi(t, \xi, \mathbf{y}) = K$$

and hence

$$\begin{aligned} & w_t - \varepsilon w_{xx} - \varepsilon w_{\xi\xi} - \Delta_{\mathbf{y}} w|_N \\ & \quad = -K\bigl(g\bigl(t, u(t, x, \mathbf{y})\bigr) - g\bigl(t, u(t, \xi, \mathbf{y})\bigr)\bigr) - \bigl(f\bigl(t, u(t, x, \mathbf{y})\bigr) - f\bigl(t, u(t, \xi, \mathbf{y})\bigr)\bigr)\big|_N. \end{aligned}$$

Due to (2.4),

$$w_t - \varepsilon w_{xx} - \varepsilon w_{\xi\xi} - \Delta_{\mathbf{y}} w|_N < 0$$

which is impossible. Let us show that $w \leqslant 0$ on Γ_P. By definition

$$\Gamma_P = \bigcup_{i=1}^{2n+4} \Omega_i,$$

where

$$\Omega_1 = \{t = 0, -l \leqslant \xi < x \leqslant l, |y_i| \leqslant l_i, i = 1, \ldots, n\},$$
$$\Omega_2 = \{0 < t \leqslant T, \xi = -l, -l \leqslant x \leqslant l, |y_i| \leqslant l_i, i = 1, \ldots, n\},$$
$$\Omega_3 = \{0 < t \leqslant T, -l \leqslant \xi \leqslant l, x = l, |y_i| \leqslant l_i, i = 1, \ldots, n\},$$
$$\Omega_4 = \{0 < t \leqslant T, \xi = x, -l \leqslant x \leqslant l, |y_i| \leqslant l_i, i = 1, \ldots, n\},$$
$$\Omega_5 = \{0 < t \leqslant T, -l \leqslant \xi < x \leqslant l, y_1 = l_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},$$
$$\vdots$$
$$\Omega_{n+4} = \{0 < t \leqslant T, -l \leqslant \xi < x \leqslant l, |y_i| \leqslant l_i, i = 1, \ldots, n-1, y_n = l_n\},$$
$$\Omega_{n+5} = \{0 < t \leqslant T, -l \leqslant \xi < x \leqslant l, y_1 = -l_1 |y_i| \leqslant l_i, i = 2, \ldots, n\},$$
$$\vdots$$
$$\Omega_{2n+4} = \{0 < t \leqslant T, -l \leqslant \xi < x \leqslant l, |y_i| \leqslant l_i, i = 1, \ldots, n-1, y_n = -l_n\}.$$

In Ω_1 we have (see (1.4))

$$w = u_0(x, \mathbf{y}) - u_0(\xi, \mathbf{y}) - K(x - \xi) \leqslant 0.$$

From Lemma 2.2, in Ω_2 we have

$$w = u(t, x, \mathbf{y}) - K(x + l) \leqslant 0$$

and in Ω_3 we have

$$w = -u(t, \xi, \mathbf{y}) - K(l - \xi) \leqslant 0.$$

In Ω_4 obviously $w = 0$. Finally, in Ω_{k+4} for $k = 1, \ldots, 2n$ we have

$$w = -K(x - \xi) \leqslant 0.$$

Thus

$$w(t, x, \xi, \mathbf{y}) \leqslant 0 \quad \text{in } \overline{P}. \tag{2.7}$$

Now subtracting Eq. (2.5) from (2.6) for the function

$$v(t, x, \xi, \mathbf{y}) \equiv u(t, \xi, \mathbf{y}) - u(t, x, \mathbf{y}) - K(x - \xi)$$

we obtain

$$v_t - \varepsilon v_{xx} - \varepsilon v_{\xi\xi} - \Delta_{\mathbf{y}} v = -\big(g\big(t, u(t,\xi,\mathbf{y})\big)u_\xi(t,\xi,\mathbf{y}) - g\big(t, u(t,x,\mathbf{y})\big)u_x(t,x,\mathbf{y})\big) \\ - \big(f\big(t, u(t,\xi,\mathbf{y})\big) - f\big(t, u(t,x,\mathbf{y})\big)\big).$$

Consider this equation in the domain P. Suppose that function v attains its positive maximum at some point $N_1 \in P \setminus \Gamma_P$, then at this point we have

$$v > 0, \qquad v_x = v_\xi = 0, \quad \text{i.e. } u(t,\xi,\mathbf{y}) > u(t,x,\mathbf{y}), u_x(t,x,\mathbf{y}) = u_\xi(t,\xi,\mathbf{y}) = -K$$

and hence

$$v_t - \varepsilon v_{xx} - \varepsilon v_{\xi\xi} - \Delta_{\mathbf{y}} v|_N \\ = K\big(g\big(t, u(t,\xi,\mathbf{y})\big) - g\big(t, u(t,x,\mathbf{y})\big)\big) - \big(f\big(t, u(t,\xi,\mathbf{y})\big) - f\big(t, u(t,x,\mathbf{y})\big)\big)\big|_N.$$

Thus, from (2.4),

$$v_t - \varepsilon v_{xx} - \varepsilon v_{\xi\xi} - \Delta_{\mathbf{y}} w|_N < 0,$$

which is impossible.

Similarly to the previous case we can show that $v \leqslant 0$ on Γ_P. Thus

$$v(t,x,\xi,\mathbf{y}) \leqslant 0 \quad \text{in } \overline{P}. \tag{2.8}$$

From (2.7) and (2.8) it follows that for $t \in [0,T]$, $-l \leqslant \xi \leqslant x \leqslant l$, $|y_i| \leqslant l_i$ the inequality

$$\big|u(t,x,\mathbf{y}) - u(t,\xi,\mathbf{y})\big| \leqslant K(x - \xi)$$

holds. Due to the symmetry of the variables x and ξ we can similarly consider the case $\xi > x$ to obtain

$$\big|u(t,x,\mathbf{y}) - u(t,\xi,\mathbf{y})\big| \leqslant K(\xi - x)$$

for $t \in [0,T]$, $-l \leqslant x \leqslant \xi \leqslant l$, $|y_i| \leqslant l_i$. Thus for $t \in [0,T]$, $x, \xi \in [-l, l]$ and $|y_i| \leqslant l_i$ we have

$$\big|u(t,x,\mathbf{y}) - u(t,\xi,\mathbf{y})\big| \leqslant K|\xi - x|,$$

which implies the needed estimate. □

Lemma 2.5. *Let $u(t,x,\mathbf{y})$ be a classical solution of problem* (2.1)–(2.3) *and assume that condition* (2.4) *is fulfilled, then*

$$\big|u_{y_i}(t,x,\mathbf{y})\big| \leqslant K_i, \quad i = 1, \ldots, n, \ \textit{in } \bar{S}_T^l.$$

Proof. We will prove the lemma for $i = 1$, the proof for $i > 1$ is similar. Consider Eq. (2.1) in two different points $(t, x, \mathbf{y})$ and $(t, x, \mathbf{y}')$, where $\mathbf{y}' = (\eta, y_2, \dots, y_n)$:

$$u_t + g(t, u)u_x + f(t, u) = \varepsilon u_{xx} + \Delta_{\mathbf{y}} u, \quad u = u(t, x, \mathbf{y}), \tag{2.9}$$

$$u_t + g(t, u)u_x + f(t, u) = \varepsilon u_{xx} + \Delta_{\mathbf{y}'} u, \quad u = u\big(t, x, \mathbf{y}'\big). \tag{2.10}$$

Subtracting Eq. (2.10) from (2.9) for the function

$$\omega\big(t, x, y, \mathbf{y}'\big) \equiv u(t, x, \mathbf{y}) - u\big(t, x, \mathbf{y}'\big) - K_1(y_1 - \eta)$$

we obtain

$$\begin{aligned}
&\omega_t - \varepsilon\omega_{xx} - \Delta_{\mathbf{y}}\omega - \omega_{\eta\eta} \\
&\quad = -\big(g\big(t, u(t, x, \mathbf{y})\big)u_x(t, x, \mathbf{y}) - g\big(t, u\big(t, x, \mathbf{y}'\big)\big)u_x\big(t, x, \mathbf{y}'\big)\big) \\
&\qquad - \big(f\big(t, u(t, x, \mathbf{y})\big) - f\big(t, u\big(t, x, \mathbf{y}'\big)\big)\big).
\end{aligned}$$

Consider this equation in the domain

$$P_1 = \big\{\big(t, x, y_1, \mathbf{y}'\big)\colon\ 0 < t < T, |x| < l, -l_1 < \eta < y_1 < l_1, |y_i| < l_i, i = 2, \dots, n\big\}.$$

Denote by Γ_{P_1} the parabolic boundary of P_1. Suppose that function ω attains its positive maximum at some point $N \in P_1 \setminus \Gamma_{P_1}$, then at this point we have

$$\omega > 0, \qquad \omega_x = 0, \quad \text{i.e. } u(t, x, \mathbf{y}) > u\big(t, x, \mathbf{y}'\big), u_x(t, x, \mathbf{y}) = u_x\big(t, x, \mathbf{y}'\big)$$

and hence taking into account the previous Lemma we obtain

$$\begin{aligned}
&\omega_t - \varepsilon\omega_{xx} - \Delta_{\mathbf{y}}\omega - \omega_{\eta\eta}|_N \\
&\quad = -u_x(t, x, \mathbf{y})\big(g\big(t, u(t, x, \mathbf{y})\big) - g\big(t, u\big(t, x, \mathbf{y}'\big)\big)\big) - \big(f\big(t, u(t, x, \mathbf{y})\big) - f\big(t, u\big(t, x, \mathbf{y}'\big)\big)\big)\big|_N \\
&\quad \leqslant K\big|g\big(t, u(t, x, \mathbf{y})\big) - g\big(t, u\big(t, x, \mathbf{y}'\big)\big)\big| - \big(f\big(t, u(t, x, \mathbf{y})\big) - f\big(t, u\big(t, x, \mathbf{y}'\big)\big)\big)\big|_N.
\end{aligned}$$

Due to (2.4) we have

$$\omega_t - \varepsilon\omega_{xx} - \Delta_{\mathbf{y}}\omega - \omega_{\eta\eta}|_N < 0,$$

which is impossible. Let us show that $\omega \leqslant 0$ on Γ_{P_1}. By definition

$$\Gamma_{P_1} = \bigcup_{i=1}^{2n+4} D_i,$$

where

$$
\begin{aligned}
&D_1 = \{t = 0, |x| \leqslant l, -l_1 \leqslant \eta < y_1 \leqslant l_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},\\
&D_2 = \{0 < t \leqslant T, |x| \leqslant l, \eta = -l_1, |y_i| \leqslant l_i, i = 1, \ldots, n\},\\
&D_3 = \{0 < t \leqslant T, |x| \leqslant l, -l_1 \leqslant \eta \leqslant l_1, y_1 = l_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},\\
&D_4 = \{0 < t \leqslant T, |x| \leqslant l, \eta = y_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},\\
&D_5 = \{0 < t \leqslant T, x = l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},\\
&D_6 = \{0 < t \leqslant T, |x| \leqslant l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, y_2 = l_2, |y_i| \leqslant l_i, i = 3, \ldots, n\},\\
&\vdots\\
&D_{n+4} = \{0 < t \leqslant T, |x| \leqslant l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, |y_i| \leqslant l_i, i = 2, \ldots, n-1, y_n = l_n\},\\
&D_{n+5} = \{0 < t \leqslant T, x = -l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, |y_i| \leqslant l_i, i = 2, \ldots, n\},\\
&D_{n+6} = \{0 < t \leqslant T, |x| \leqslant l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, y_2 = -l_2, |y_i| \leqslant l_i, i = 3, \ldots, n\},\\
&\vdots\\
&D_{2n+4} = \{0 < t \leqslant T, |x| \leqslant l, -l_1 \leqslant \eta \leqslant y_1 \leqslant l_1, |y_i| \leqslant l_i, i = 2, \ldots, n-1, y_n = -l_n\}.
\end{aligned}
$$

In D_1 we have

$$\omega = u_0(x, \mathbf{y}) - u_0(x, \mathbf{y}') - K_1(y_1 - \eta) \leqslant 0,$$

in D_2 and D_3 due to the boundary conditions and Lemma 2.3 we have

$$\omega = u(t, x, \mathbf{y}) - K_1(y_1 + l_1) \leqslant 0$$

and

$$\omega = -u(t, x, \mathbf{y}') - K_1(l_1 - \eta) \leqslant 0,$$

respectively. On D_4 obviously

$$\omega = 0,$$

respectively. On D_i, $i = 5, \ldots, 2n+4$, due to the boundary conditions

$$\omega = -K_1(y_1 - \eta) \leqslant 0.$$

Hence

$$\omega \leqslant K_1(y_1 - \eta) \quad \text{in } \overline{P_1}. \tag{2.11}$$

Now subtracting Eq. (2.9) from (2.10) for the function

$$\mathrm{v}\big(t,x,y,\mathbf{y}'\big) \equiv u\big(t,x,\mathbf{y}'\big) - u(t,x,\mathbf{y}) - K_1(y_1-\eta)$$

we obtain

$$\begin{aligned}&\mathrm{v}_t - \varepsilon \mathrm{v}_{xx} - \Delta_{\mathbf{y}}\mathrm{v} - \mathrm{v}_{\eta\eta}\\ &\quad = -\big(g\big(t,u\big(t,x,\mathbf{y}'\big)\big)u_x\big(t,x,\mathbf{y}'\big) - g\big(t,u(t,x,\mathbf{y})\big)u_x(t,x,\mathbf{y})\big)\\ &\qquad - \big(f\big(t,u\big(t,x,\mathbf{y}'\big)\big) - f\big(t,u(t,x,\mathbf{y})\big)\big).\end{aligned}$$

Suppose that function v attains its positive maximum at some point $N \in P_1 \setminus \varGamma_{P_1}$, then at this point we have

$$\mathrm{v} > 0, \qquad \mathrm{v}_x = 0, \quad \text{i.e. } u\big(t,x,\mathbf{y}'\big) > u(t,x,\mathbf{y}), u_x(t,x,\mathbf{y}) = u_x\big(t,x,\mathbf{y}'\big)$$

and hence

$$\begin{aligned}&\mathrm{v}_t - \varepsilon \mathrm{v}_{xx} - \Delta_{\mathbf{y}}\mathrm{v} - \mathrm{v}_{\eta\eta}|_N\\ &\quad = -u_x(t,x,\mathbf{y})\big(g\big(t,u\big(t,x,\mathbf{y}'\big)\big) - g\big(t,u(t,x,\mathbf{y})\big)\big) - \big(f\big(t,u\big(t,x,\mathbf{y}'\big)\big) - f\big(t,u(t,x,\mathbf{y})\big)\big)\big|_N\\ &\quad \leqslant K\big|g\big(t,u\big(t,x,\mathbf{y}'\big)\big) - g\big(t,u(t,x,\mathbf{y})\big)\big| - \big(f\big(t,u\big(t,x,\mathbf{y}'\big)\big) - f\big(t,u(t,x,\mathbf{y})\big)\big)\big|_N.\end{aligned}$$

Due to (2.4) we have

$$\mathrm{v}_t - \varepsilon \mathrm{v}_{xx} - \mathrm{v}_{yy} - \mathrm{v}_{\mathbf{y}'\mathbf{y}'}|_N < 0,$$

which is impossible.

Similarly to the previous case we can show that $\mathrm{v} \leqslant 0$ on $\varGamma_{P_1}$ and consequently

$$\mathrm{v} \leqslant K_1(y_1-\eta) \quad \text{in } \overline{P_1}. \tag{2.12}$$

From (2.10) and (2.11) we conclude that

$$\big|u(t,x,\mathbf{y}) - u\big(t,x,\mathbf{y}'\big)\big| \leqslant K_1(y_1-\eta) \quad \text{in } \overline{P_1}.$$

Due to the symmetry of the variables y_1 and η we can similarly consider the case $\eta > y_1$ to obtain

$$\big|u(t,x,\mathbf{y}) - u\big(t,x,\mathbf{y}'\big)\big| \leqslant K_1(\eta-y_1)$$

for $t \in [0,T]$, $|x| \leqslant l$, $-l_1 \leqslant y_1 \leqslant \eta \leqslant l_1$ and $|y_i| \leqslant l_i$, $i = 2,\ldots,n$.

Finally, for $t \in [0,T]$, $|x| \leqslant l$, $y_1,\eta \in [-l_1,l_1]$ and $|y_i| \leqslant l_i$, $i = 2,\ldots,n$, we have

$$\big|u(t,x,\mathbf{y}) - u\big(t,x,\mathbf{y}'\big)\big| \leqslant K_1|y_1-\eta|$$

which implies the needed estimate. □

Let us extend Lemmas 1–5 to the case when non strict inequality (1.3) instead of strict (2.4) takes place.

Lemma 2.6. *Lemmas* 2.1–2.5 *remain correct when substituting condition* (2.4) *by* (1.3).

Proof. Suppose that condition (1.3) is fulfilled. Consider the function $v = u\mathrm{e}^{-\epsilon t}$ satisfying the equation

$$v_t + g\big(t, v\mathrm{e}^{\epsilon t}\big)v_x + \tilde{f}\big(\big(t, v\mathrm{e}^{\epsilon t}\big)\big) = \Delta_{\mathbf{y}} v + \varepsilon v_{xx}$$

and the initial-boundary conditions

$$v|_{t=0} = u_0, \qquad v|_{x=\pm l} = v|_{y_i=\pm l_i} = 0, \quad i = 1, \ldots, n,$$

where $\tilde{f}((t, v\mathrm{e}^{\epsilon t})) = f((t, v\mathrm{e}^{\epsilon t})) + \epsilon v$. Similarly to the proofs of Lemmas 2.1–2.3 we prove the estimates

$$|v| \leqslant M, \qquad |v| \leqslant K(l + x), \qquad |v| \leqslant K(l - x), \qquad |v| \leqslant K_i(l_i + y_i), \qquad |v| \leqslant K_i(l_i - y_i),$$
$$i = 1, \ldots, n,$$

and passing to the limit when $\epsilon \to 0$ we obtain the needed estimates for u.

Since condition (1.3) is fulfilled the following inequality takes place

$$K\big|g\big(t, v_2\mathrm{e}^{\epsilon t}\big) - g\big(t, v_1\mathrm{e}^{\epsilon t}\big)\big| < \tilde{f}\big(t, v_2\mathrm{e}^{\epsilon t}\big) - \tilde{f}\big(t, v_1\mathrm{e}^{\epsilon t}\big) \quad \text{for } v_2 > v_1. \tag{2.13}$$

Similarly to the proofs of Lemmas 2.4 and 2.5 we obtain the estimates

$$|v_x| \leqslant K, \qquad |v_{y_i}| \leqslant K_i, \quad i = 1, \ldots, n,$$

and passing to the limit while $\epsilon \to 0$ we obtain the needed estimate. □

Remark. When proving the estimate for v_x (v_{y_i}) we consider equation in two different points $(t, x, \mathbf{y})$ and $(t, \xi, \mathbf{y})$ ($(t, x, \mathbf{y})$, $(t, x, \mathbf{y}')$), i.e. variable t remains the same, consequently we require the fulfillment of (2.13) and *not* of

$$K\big|g\big(t, v_2\mathrm{e}^{\epsilon t_2}\big) - g\big(t, v_1\mathrm{e}^{\epsilon t_1}\big)\big| < \tilde{f}\big(t, v_2\mathrm{e}^{\epsilon t_2}\big) - \tilde{f}\big(t, v_1\mathrm{e}^{\epsilon t_1}\big).$$

In the next lemma we prove the L_2 estimates of u and u_{y_i} independent both of ε and of the size of the domain Q^l.

Lemma 2.7. *For any classical solution of problem* (2.1)–(2.3) *we have*

$$\int_{Q^l} u^2 \,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \int_{S_T^l} \sum_{i=1}^{n} u_{y_i}^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y} \leqslant U_0,$$

where

$$U_0 = \int_{Q^l} u_0^2(x, \mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}x.$$

Proof. Multiply Eq. (2.1) by u and integrate by part with respect to $\mathbf{y}$ and to x we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{Q^l} u^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \int_{Q^l}\sum_{i=1}^{n} u_{y_i}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x \leqslant 0$$

from where integrating with respect to t we obtain the needed estimate. We use here the inequality $[g(t,u)u_x + f(t,u)]u \geqslant 0$ which follows immediately from (1.3). $\Box$

We close this section by the following lemma where we prove the L_2 estimates of u_t and $u_{y_iy_j}$ independent of ε but depending on the size of the domain Q^l.

Lemma 2.8. *For any classical solution of problem* (2.1)–(2.3) *we have*

$$\int_{S_T^l} u_t^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y} \leqslant C_0, \qquad \int_{S_T^l} u_{y_iy_j}^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y} \leqslant C_{ij}, \quad i,j = 1,\ldots,n,$$

where the constants C_0, C_{ij} are independent of ε.

Proof. Multiply Eq. (2.1) by u_t and integrate by part with respect to $\mathbf{y}$ and to x we obtain:

$$\begin{aligned}
&\int_{Q^l} u_t^2 d\mathbf{y}\,\mathrm{d}x + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{Q^l} u_x^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{Q^l}\sum_{i=1}^{n} u_{y_i}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad = -\int_{Q^l} u_t\big[g(t,u)u_x + f(t,u)\big]\,\mathrm{d}\mathbf{y}\,\mathrm{d}x \leqslant \frac{1}{2}\int_{Q^l} u_t^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \frac{1}{2}\int_{Q^l}\big[g(t,u)u_x + f(t,u)\big]^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x,
\end{aligned}$$

where $\mathrm{d}\mathbf{y} = \mathrm{d}y_1\cdots\mathrm{d}y_n$. Now integrating with respect to t and taking into account Lemma 2.1 and Lemma 2.4 we obtain the first estimate.

In order to obtain the remaining estimates multiply Eq. (2.1) by $u_{y_1y_1}$ and integrate by part with respect to $\mathbf{y}$ and x, we obtain:

$$\begin{aligned}
&\int_{Q^l} u_{y_1y_1}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \varepsilon\int_{Q^l} u_{xy_1}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{Q^l} u_{y_1}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \int_{Q^l}\sum_{j=2}^{n} u_{y_1y_j}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad = \int_{Q^l}\big[g(t,u)u_x + f(t,u)\big]u_{y_1y_1}\,\mathrm{d}\mathbf{y}\,\mathrm{d}x \leqslant \frac{1}{2}\int_{Q^l}\big[g(t,u)u_x + f(t,u)\big]^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x + \frac{1}{2}\int_{Q^l} u_{y_1y_1}^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x.
\end{aligned}$$

Integrating with respect to t and taking into account Lemmas 2.1 and 2.4 we obtain the second estimate for $i = 1$, similarly we can consider the case $i > 1$. $\Box$

3. Proof of Theorems 1–4

We start from the following proof.

Proof of Theorem 3. Assumptions of Theorem 3 guarantee the existence of a classical solution of problem (2.1)–(2.3) (see [19]). Actually the classical solvability follows from the Lemmas 2.1, 2.5, 2.6 and the following estimate (see, e.g., [17]):

$$\left|u^\varepsilon(t_1,x,\mathbf{y})-u^\varepsilon(t_2,x,\mathbf{y})\right| \leqslant K_0|t_1-t_2|^{1/2},$$

where the constant K_0 depends only on $M=\max|u^\varepsilon|$, $K=\max|u_x^\varepsilon|$, $K_i=\max|u_{y_i}^\varepsilon|$, $i=1,\dots,n$, and does not depend on ε^{-1}.

From the above estimate and the estimates obtained in previous sections it follows that we can find a subsequence ε_k such that for $k\to+\infty$ ($\varepsilon_k\to 0$)

$$u^{\varepsilon_k} \Longrightarrow u \quad \text{uniformly},$$

hence the limit function u is Hölder continuous and

$$\left|u(t_1,x,\mathbf{y})-u(t_2,x,\mathbf{y})\right| \leqslant K_0|t_1-t_2|^{1/2}.$$

Moreover,

$$\begin{aligned}
&u_x^{\varepsilon_k}\to u_x, \qquad u_{y_i}^{\varepsilon_k}\to u_{y_i}, \qquad i=1,\dots,n, \ {}^*\text{weakly in } L_\infty\bigl(S_T^l\bigr),\\
&u_t^{\varepsilon_k}\to u_t, \qquad u_{y_iy_i}^{\varepsilon_k}\to u_{y_iy_i}, \quad i=1,\dots,n, \ \text{weakly in } L_2\bigl(Q_T^l\bigr).
\end{aligned}$$

Consider the following integral identity

$$\int_{S_T^l}\bigl[u_t^{\varepsilon_k}+g\bigl(t,u^{\varepsilon_k}\bigr)u_x^{\varepsilon_k}+f\bigl(t,u^{\varepsilon}\bigr)-\Delta_{\mathbf{y}}u^{\varepsilon_k}\bigr]\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y}=-\varepsilon\int_{S_T^l}u_x^{\varepsilon_k}\phi_x\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y},$$

where $\phi,\phi_x\in L^2(S_T^l)$. Passing to the limit when $k\to+\infty$ we obtained the required solution:

$$\int_{S_T^l}\bigl[u_t+g(t,u)u_x+f(t,u)-\Delta_{\mathbf{y}}u\bigr]\phi\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mathbf{y}=0$$

$\forall\phi\in L^2(S_T^l)$ such that $\phi_x\in L^2(S_T^l)$.

In order to prove the uniqueness we suppose that there exists two solutions u_1 and u_2. For the difference $\tilde u\equiv u_1-u_2$ we obtain

$$\tilde u_t+g(t,u_1)\tilde u_x+\bigl(g(t,u_1)-g(t,u_2)\bigr)u_{2x}+f(t,u_1)-f(t,u_2)=\Delta_{\mathbf{y}}\tilde u \quad \text{a.e. in } S_T^l.$$

Multiply this relation by $\tilde u$ and integrate by part to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Q^l}\tilde u^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x+\int_{Q^l}|\nabla_{\mathbf{y}}\tilde u|^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\qquad+\int_{Q^l}\bigl[\bigl(g(t,u_1)-g(t,u_2)\bigr)u_{2x}+f(t,u_1)-f(t,u_2)\bigr](u_1-u_2)\,\mathrm{d}\mathbf{y}\,\mathrm{d}x\\
&\quad=\frac{1}{2}\int_{Q^l}g_{u_1}(t,u_1)u_{1x}\tilde u^2\,\mathrm{d}\mathbf{y}\,\mathrm{d}x.
\end{aligned}$$

Integrate with respect to t. Taking into account (2.4) and the fact that $\tilde{u}(0, x, y) \equiv 0$ we obtain

$$\int_{Q^l} \tilde{u}^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}x \leqslant \int_{S_T^l} g_{u_1}(t, u_1) u_{1x} \tilde{u}^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}x.$$

Apply Gronwall's inequality to obtain

$$\int_{Q^l} \tilde{u}^2 \, \mathrm{d}\mathbf{y} \, \mathrm{d}x = 0 \quad \Longleftrightarrow \quad \tilde{u} \equiv 0. \qquad \square$$

Proof of Theorem 4. In order to prove Theorem 4 introduce the function $v = u \, \mathrm{e}^{-\mu t}$ (with μ – positive constant) which satisfies the equation

$$v_t + g\big(t, v\mathrm{e}^{\mu t}\big) v_x + \tilde{f}\big(t, v\mathrm{e}^{\mu t}\big) = \Delta_{\mathbf{y}} v \quad \text{in } S_T$$

and the initial condition

$$v|_{t=0} = u_0,$$

here $\tilde{f}(t, v\mathrm{e}^{\mu t}) = f(t, v\mathrm{e}^{\mu t}) + \mu v$. Condition (1.3) takes the form (see remark to Lemma 2.6)

$$K\big|g\big(t, v_2\mathrm{e}^{\mu t}\big) - g\big(t, v_1\mathrm{e}^{\mu t}\big)\big| \leqslant f\big(t, v_2\mathrm{e}^{\mu t}\big) - f\big(t, v_1\mathrm{e}^{\mu t}\big) + \mu(v_2 - v_1) \quad \text{for } v_1 < v_2. \tag{3.1}$$

Denote by g_{L} and f_{L} the Lipschitz constant of g and f, respectively. Taking into account the assumptions of Theorem 4 for any $\mu > Kg_{\mathrm{L}} + f_{\mathrm{L}}$ we can find $T^* > 0$ small enough such that condition (3.1) is fulfilled for all $t \in (0, T^*)$ and as a consequence we obtain the existence of a generalized solution on the interval $(0, T^*)$ (see Example 5). Note that due to the condition $uf(t, u) \geqslant 0$ the solution is bounded and as a consequence g_{L} and f_{L} are bounded. If condition $uf(t, u) \geqslant 0$ is not fulfilled then we can find $T^* > 0$ small enough such that condition (3.1) is fulfilled for all $t \in (0, T^*)$ under the assumption that g and f are global Lipschitz continuous functions (see Example 6). $\square$

The following arguments explain why we can impose two boundary conditions in the x direction while only the first-order derivative with respect to x is present in the equation. Consider the following problem

$$u_t + g(t, u) u_x + f(t, u) = 0 \quad \text{in } S_T, \qquad u|_{t=0} = u_0(x), \quad x \in \mathbf{R}. \tag{3.2}$$

Problem (3.2) is equivalent to the following one

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = g(t, u), & x(0) = \xi, \\ \mathrm{d}u/\mathrm{d}t = -f(t, u), & u(0) = u_0(\xi). \end{cases} \tag{3.3}$$

Suppose that $u_0(\pm l) = 0$. The characteristics of system (3.3) starting from $x(0) = \pm l$ are $x(t) = \pm l$ and the function u along these characteristics is equal to zero (recall that $g(t, 0) = f(t, 0) = 0$), i.e. the solution of problem (3.2) satisfies condition $u(t, \pm l) = 0$. Obviously the solution of problem (3.2) is at the same time solution of the following one

$$u_t + g(t, u) u_x + f(t, u) = \Delta_{\mathbf{y}} u \quad \text{in } S_T, u|_{t=0} = u_0(x), x \in \mathbf{R}. \tag{3.4}$$

Actually Eq. (1.8) and the fact that $u_0|_{x=\pm l} = 0$ create the boundary conditions $u|_{x=\pm l} = 0$.

Proof of Theorem 1. Denote by $u_0^l(x, \mathbf{y})$, $l = 1, 2, \dots$, a sequence of Lipschitz continuous functions such that

$$u_0^l\big|_{x=\pm l} = u_0^l\big|_{y_i=\pm l} = 0, \quad i = 1, \dots, n,$$

and which approximate the initial function u_0 from (1.2). Consider the following problem

$$u_t + g(t, u)u_x + f(t, u) = \Delta_{\mathbf{y}} u \quad \text{in } (0, T) \times (-l, l)^{n+1},$$
$$u|_{t=0} = u_0^l(x, \mathbf{y}) \quad \text{in } (-l, l)^{n+1}, \qquad u|_{x=\pm l} = u|_{y_i=\pm l} = 0, \quad i = 1, \dots, n.$$

The solution of the Cauchy problem now can be obtained as the limit of a sequence of solutions of the above problem when $l \to +\infty$. For more details see [8]. $\square$

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 4. $\square$

Let us give two examples to demonstrate Theorem 2 and 4. We give these examples for the Cauchy problem but they can be trivially extended to the initial-boundary value problem.

Example 5. Consider the equation

$$u_t + g(t, u)u_x + \lambda g(t, u) = \Delta_{\mathbf{y}} u, \quad \lambda \geqslant 0. \tag{3.5}$$

Assume that g is locally Lipschitz continuous and nondecreasing with respect to u function such that $g(t, 0) = 0$ (note that these imply inequality $ug(t, u) \geqslant 0$). For the function $v = u\mathrm{e}^{-\mu t}$ problem (3.5), (1.2) takes the following form

$$v_t + g\big(t, v\mathrm{e}^{\mu t}\big)v_x + \lambda g\big(t, v\mathrm{e}^{\mu t}\big) + \mu v = \Delta_{\mathbf{y}} v \quad \text{in } S_T, v|_{t=0} = u_0 \tag{3.6}$$

and condition (1.3) takes the form

$$(K - \lambda)\big(g\big(t, v_2\mathrm{e}^{\mu t}\big) - g\big(t, v_1\mathrm{e}^{\mu t}\big)\big) \leqslant \mu(v_2 - v_1) \quad \text{for } v_2 > v_1. \tag{3.7}$$

These inequality is fulfilled if

$$(K - \lambda)\mathrm{e}^{\mu t} g_{\mathrm{L}} \leqslant \mu.$$

Here g_{L} is a Lipschitz constant for the function g, taking into account that for the solution v of problem (3.6) the estimate $|v| \leqslant m = \max|u_0|$ takes place we conclude that $g_{\mathrm{L}} < \infty$. Obviously if $K \leqslant \lambda$ then condition (3.7) is fulfilled for arbitrary t and as a consequence we have global solvability (see Example 1 in the Introduction). Let $K > \lambda$, we can define T^* from the relation

$$(K - \lambda)\mathrm{e}^{\mu T^*} g_{\mathrm{L}} = \mu.$$

One can easily see that the maximal value of T^* satisfying this relation is

$$T^* = \sup_{\mu > C_0} \frac{1}{\mu} \ln \frac{\mu}{C_0} = \frac{1}{\mu} \ln \frac{\mu}{C_0}\bigg|_{\mu = C_0 e} = \frac{1}{C_0 e},$$

here $C_0 = (K - \lambda) g_{\mathrm{L}}$. Thus Theorem 4 guaranties the solvability of problem (3.5), (1.2) on the interval $(0, (C_0 e)^{-1})$. It is clear that $T^* \to +\infty$ when $\lambda \to K$.

Example 6. Consider equation

$$u_t + a\, u\, u_x + \lambda u = \Delta_{\mathbf{y}} u, \quad a, \lambda \in \mathbf{R}. \tag{3.8}$$

For the function $v = u\, \mathrm{e}^{-\mu t}$ problem (3.8), (1.2) takes the following form

$$v_t + a \mathrm{e}^{\mu t} v v_x + (\lambda + \mu) v = \Delta_{\mathbf{y}} v \quad \text{in } S_T, v|_{t=0} = u_0 \tag{3.9}$$

while condition (1.3) takes the form

$$K|a| \mathrm{e}^{\mu t} \leqslant \lambda + \mu. \tag{3.10}$$

If $\lambda \geqslant K|a|$ then we can take $\mu = 0$ and (3.10) is fulfilled for arbitrary $t > 0$ guaranteeing the global solvability of problem (3.8), (1.2) (see Example 4 in the Introduction). If $\lambda < |a|\, K$ then for any $\mu > K|a| - \lambda$ there exists T^* such that for $t \in [0, T^*)$ (3.10) is satisfied, one can easily see that T^* is defined from the relation

$$T^* = \sup_{\mu > K|a| - \lambda} \frac{1}{\mu} \ln \frac{\lambda + \mu}{K|a|}.$$

It is clear that if $\lambda = 0$ then $T^* = (eK|a|)^{-1}$ and if $\lambda > K|a|$ then $T^* = +\infty$. Moreover, $T^* \to +\infty$ for $K \to 0$.

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