Arch. Math. 88 (2007), 259–268

1422-6928/030259-10, *published online* 2007-02-19
DOI 10.1007/s00013-006-1873-9

Archiv der Mathematik

Viscosity solutions of the p-Laplace equation with nonlinear source

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Abstract. In the present paper we consider the Dirichlet problem in a convex domain for the multidimensional p-Laplace equation with nonlinear source. We prove the existence of the unique continuous viscosity solution under quite general assumptions on the structure of the source.

Mathematics Subject Classification (2000). 35J25, 35J70.

Keywords. Viscosity solutions, p-Laplace equation.

0. Introduction. Consider the p-Laplace equation

$$(0.1) \qquad -\text{div}(|\nabla u|^{p-2}\nabla u) + g(\mathbf{x}, u, \nabla u) + f(\mathbf{x}) = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n,\ p \geq 2,$$

coupled with the Dirichlet boundary condition

$$(0.2) \qquad u = 0 \quad \text{for} \quad \mathbf{x} \in \partial\Omega,$$

where we suppose that Ω is a strictly convex domain lying in the parallelepiped

$$\Omega \subset \{\mathbf{x} = (x_1, \ldots, x_n) : -l_i \leq x_i \leq l_i, \quad i = 1, ..., n\}.$$

We assume that the parts of $\partial\Omega$ lying in the half-spaces $x_1 \leq 0$ and $x_1 \geq 0$ can be expressed as

$$x_1 = F(x_2, x_3, ..., x_n), \quad x_1 = G(x_2, x_3, ..., x_n)$$

respectively.

In this paper we focus our attention on the problem of construction of continuous viscosity sub and supersolutions for equation (0.1) that satisfy the boundary condition (0.2). Let us recall the precise meaning of the notion of continuous viscosity solution of (0.1), (0.2). According to [7], $u(\mathbf{x})$ is a continuous viscosity subsolution of (0.1), (0.2), if $u(\mathbf{x})$ is upper semicontinuous function on Ω and for every $\phi(\mathbf{x}) \in C^2(\Omega)$ and local maximum point $\hat{\mathbf{x}} \in \Omega$ of $u - \phi$, one has

$$\Phi(\hat{\mathbf{x}}, u(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}})) \leq 0,$$

where

(0.3)

$$\Phi(\mathbf{x}, r, \mathbf{q}, X) = -|\mathbf{q}|^{p-2} Tr(X) - (p-2)|\mathbf{q}|^{p-4} \mathrm{Tr}((\mathbf{q} \otimes \mathbf{q})X) + g(\mathbf{x}, r, \mathbf{q}) + f(\mathbf{x}).$$

The notion of a continuous viscosity supersolution of (0.1), (0.2) arises replacing "upper semicontinuous" by "lower semicontinuous", "max" by "min" and reversing the inequality to

$$\Phi(\hat{\mathbf{x}}, u(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), D^2\phi(\hat{\mathbf{x}})) \geq 0.$$

Continuous viscosity solution of (0.1), (0.2) is a function which is simultaneously a continuous viscosity subsolution and a continuous viscosity supersolution. The boundary condition is interpreted in the strict sense.

Viscosity solutions were introduced by M.G. Crandall and P.L. Lions [3] in the case of Hamilton-Jacobi equations and were extended to the case of second order elliptic equations by H. Ishii and P. L. Lions [7] and L. Caffarelli, M. Crandall, M. Kocan and A. Swięch [1]. The theory of viscosity solutions applies to certain partial differential equations that are proper in the sense of [5]. In the case of the p-Laplace equation (0.1) it means that the function $g(\mathbf{x}, u, \mathbf{q})$ is nondecreasing with respect to u.

It is well known that the extension of the classical maximum principle to the case of semicontinuous functions (see for example [2], [4], [6]–[14]) plays the key role in the theory of viscosity solutions. The maximum principle for semicontinuous functions gives us the comparison principle for viscosity subsolutions and supersolutions of problem (0.1), (0.2). When the comparison principle for any subsolutions and supersolutions holds one can use Ishii's implementation of Perron's method [5], [6] in order to prove the existence of continuous viscosity solution for (0.1), (0.2).

In order to use the Ishii-Perron method [5, Theorem 4.1] one has to produce a subsolution and a supersolution that vanish on the boundary. The mentioned above Theorem leaves open the question when such a subsolution and supersolution can be found. Concerning the parabolic analogue of (0.1) without the nonlinear source, we want to mention [14], where viscosity subsolutions and supersolutions were constructed for the initial problem. To the best of our knowledge there are no such results concerning the boundary value problems for p-Laplace equations with nonlinear source.

In this paper we present a new method of constructing subsolutions and supersolutions for the problem (0.1), (0.2), that one can use for a wider class of degenerate and nonuniformly elliptic equations. These subsolutions and supersolutions are solutions of certain ordinary differential equations, where the function which describes the boundary of the domain Ω plays the role of independent variable. It is worth mentioning that the subsolutions and supersolutions are given explicitly and can be used to provide modulus of continuity estimates on solutions of (0.1), (0.2).

In order to use the Ishii-Perron method one has to assume that the strong comparison result holds, which means that if u_* is a subsolution and u^* is a supersolution of $\Phi = 0$, where u_* and $-u^*$ are upper semicontinuous functions, then from $u_* \le u^*$ on $\partial\Omega$ follows that $u_* \le u^*$ in Ω. To assert that comparison holds via theorems of [5, conditions 3.13 and 3.14] one must impose structure condition on $\Phi(\mathbf{x}, r, \mathbf{q}, X)$. In the case the of p-Laplace equation, according to [5] we have that if there exists $\nu > 0$ such that

$$\nu(r-s) \le g(\mathbf{x}, r, \mathbf{q}) - g(\mathbf{x}, s, \mathbf{q}) \quad \text{for} \quad s \le r, \quad (\mathbf{x}, \mathbf{q}) \in \overline{\Omega} \times \mathbb{R}^n \tag{0.4}$$

and there is a function $\omega : [0, \infty] \to [0, \infty]$ that satisfies $\omega(0+) = 0$ such that

$$g(\mathbf{y}, r, \beta(\mathbf{x}-\mathbf{y})) - g(\mathbf{x}, r, \beta(\mathbf{x}-\mathbf{y})) + f(\mathbf{y}) - f(\mathbf{x}) \le \omega(\beta|\mathbf{x}-\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|) \tag{0.5}$$

whenever

$$\mathbf{x}, \mathbf{y} \in \Omega, \quad r \in R,$$

then the comparison principle holds. Here $\beta > 0$ is a parameter that satisfies

$$\lim_{\beta\to\infty} \beta|\mathbf{x}-\mathbf{y}|^2 = 0.$$

One may produce examples of proper Φ satisfying (0.4), (0.5) assuming, for example, that

$$g(\mathbf{x}, r, \mathbf{q}) = g_1(r, \mathbf{q}) + \nu r,$$

where g_1 is a nondecreasing function of r and $f(\mathbf{x})$ is a continuous function (with ω the modulus of continuity for f).

In order to formulate the main result of the present paper we will make some assumptions concerning the nonlinear source of (0.1). Denote by $\tilde{x} = (x_2, x_3, \ldots, x_n)$. Suppose that there exist positive constants C_0 and C_1 such that for $\rho \ge C_0$

$$-g(\mathbf{x}, 0, -\rho, \rho\nabla G(\tilde{x})) \le C_1(p-1)\rho^{p-1}, \tag{0.6}$$

$$g(\mathbf{x}, 0, \rho, -\rho\nabla G(\tilde{x})) \le C_1(p-1)\rho^{p-1}, \tag{0.7}$$

$$-g(\mathbf{x}, 0, \rho, -\rho\nabla F(\tilde{x})) \le C_1(p-1)|\rho|^{p-1}, \tag{0.8}$$

$$g(\mathbf{x}, 0, -\rho, \rho\nabla F(\tilde{x})) \le C_1(p-1)\rho^{p-1}. \tag{0.9}$$

Examples of nonlinearities g which satisfy these conditions are given at the end of the paper.

Let us formulate now the main result of the present paper.

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded strictly convex domain with $\partial\Omega \in \mathbb{C}^2$, $\Phi \in \mathbb{C}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S(n))$. Suppose that the conditions (0.4)–(0.9) are fulfilled, then there exists a unique continuous viscosity solution for the problem (0.1), (0.2).

The proof of the existence theorem is based on the Ishii-Perron method [5]. The necessary subsolutions and supersolutions of (0.1) that satisfy the boundary condition are constructed in the next section.

1. Construction of subsolutions and supersolutions. Put

$$C_0 = \left(\frac{f_0}{p-1}\right)^{\frac{1}{p-1}} > 0, \quad \text{where} \quad \max|f(\mathbf{x})| = f_0. \tag{1.1}$$

Suppose that the conditions (0.6)–(0.9) are fulfilled. Introduce the function $h(\xi)$ by

$$h(\xi) = \frac{C_0}{C_1}\left(\frac{1+C_1}{C_1}\left(e^{2C_1 l_1} - e^{C_1(2l_1-\xi)}\right) - \xi\right).$$

One can easily see that $h(\xi)$ satisfies the equation

$$h'' + C_1 h' + C_0 = 0 \tag{1.2}$$

and boundary conditions

$$h(0) = 0, \quad h(2l_1) = \frac{C_0}{C_1}\left(\frac{1+C_1}{C_1}\left(e^{2C_1 l_1} - 1\right) - 2l_1\right).$$

Obviously

$$h'(\xi) \geq C_0 \quad \text{for} \quad \xi \in [0, 2l_1],$$

thus we have that $h(\xi) \geq 0$ when $\xi \in [0, 2l_1]$.

Define the operator L by

$$Lu \equiv -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(\mathbf{x}, u, \nabla u) + f(\mathbf{x}).$$

For $\xi \equiv G(\tilde{x}) - x_1$ we have (recall that $\tilde{x} = (x_2, x_3, \ldots, x_n)$)

$$\nabla h(\xi) \equiv (h_{x_1}(\xi), h_{x_2}(\xi), \ldots, h_{x_n}(\xi)) = (-h'(\xi), h'(\xi)G_{x_2}(\tilde{x}), \ldots, h'(\xi)G_{x_n}(\tilde{x})),$$

$$|\nabla h(\xi)|^{p-2} = h'^{p-2}(\xi)(1 + |\nabla G(\tilde{x})|^2)^{\frac{p-2}{2}}.$$

Hence we have

$$Lh(\xi) = -\left(-h'^{p-1}\left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}}\right)_{x_1} - \sum_{m=2}^{n}\left(h'^{p-1}\left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} G_{x_m}\right)_{x_m}$$

$$
\begin{aligned}
&+ g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}) = -(p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} \\
&\quad - {h'}^{p-1} \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} \triangle G(\tilde{x}) \qquad (1.3) \\
&- (p-2){h'}^{p-1} \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-4}{2}} \sum_{m=2}^{n} \left(G_{x_m} \sum_{s=2}^{n} G_{x_s} G_{x_s x_m} \right) \\
&- (p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} \sum_{m=2}^{n} G^2_{x_m} + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}).
\end{aligned}
$$

Denote by $D^2G(\tilde{x})$ the matrix of second derivatives of $G(\tilde{x})$. Then we can represent the term

$$\sum_{m=2}^{n} \left(G_{x_m} \sum_{s=2}^{n} G_{x_s} G_{x_s x_m} \right)$$

in the following way

$$\sum_{m=2}^{n} \left(G_{x_m} \sum_{s=2}^{n} G_{x_s} G_{x_s x_m} \right) = \nabla G(\tilde{x}) D^2 G(\tilde{x}) (\nabla G(\tilde{x}))^T, \qquad (1.4)$$

where $(\nabla G(\tilde{x}))^T$ means the gradient vector column of G. It is well-known that (1.4) as a bilinear quadratic form is negatively defined if and only if the matrix $D^2G(\tilde{x})$ is negatively defined. And this is the case because $x_1 = G(\tilde{x})$ defines the part of $\partial\Omega$ which is convex. Furthermore, due to the convexity of the domain Ω we have $\triangle G(\tilde{x}) \leq 0$. So from (1.3) we obtain the following inequality (recall that $h'' \leq 0$)

$$
\begin{aligned}
&Lh(\xi) \geq -(p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} - \\
&(p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} \sum_{m=2}^{n} G^2_{x_m} + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}) \qquad (1.5) \\
&-(p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p-2}{2}} \left(1 + |\nabla G(\tilde{x})|^2\right) + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}) = \\
&-(p-1){h'}^{p-2}h'' \left(1 + |\nabla G(\tilde{x})|^2\right)^{\frac{p}{2}} + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}) \geq \\
&-(p-1){h'}^{p-2}h'' + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}).
\end{aligned}
$$

From (1.2) we have that $h'' = -C_1 h' - C_0$. Therefore (1.5) implies

$$
\begin{aligned}
Lh(\xi) &\geq (p-1)C_1{h'}^{p-1} + (p-1)C_0{h'}^{p-2} + g(\mathbf{x}, h, \nabla h) + f(\mathbf{x}) \\
&\geq (p-1)C_1{h'}^{p-1} + (p-1)C_0{h'}^{p-2} + g(\mathbf{x}, 0, \nabla h) + f(\mathbf{x}),
\end{aligned}
$$

where the last inequality follows from the fact that $h \geq 0$ and (0.4). Using the inequality $h' \geq C_0$ we immediately conclude

$$Lh(\xi) \geq (p-1)C_0^{p-1}(C_1 + 1) + g(\mathbf{x}, 0, \nabla h) + f(\mathbf{x}). \qquad (1.6)$$

Suppose now that $g(x, 0, \nabla h) \neq 0$. From (1.1) and (0.6) we obtain that

$$Lh(\xi) \geq (p-1)C_0^{p-1}(C_1 + 1) + g(\mathbf{x}, 0, -h'(\xi), h'(\xi)G_{x_2}, ..., h'(\xi)G_{x_n}) + f(\mathbf{x}) \geq 0.$$

If $g(x,0,\nabla h)=0$ then due to (1.1)

$$(p-1)C_0^{p-1}(C_1+1)=f_0(C_1+1)\geq f_0,$$

and

$$Lh(\xi)\geq (p-1)C_0^{p-1}(C_1+1)+f(\mathbf{x})\geq 0.$$

Consider now $h(\zeta)\equiv h(x_1-F(\tilde{x}))$. We have

$$\nabla h(\zeta)\equiv (h_{x_1}(\zeta),h_{x_2}(\zeta),\dots,h_{x_n}(\zeta))=(h'(\zeta),-h'(\zeta)F_{x_2}(\tilde{x}),\dots,-h'(\zeta)F_{x_n}(\tilde{x})),$$

$$|\nabla h(\zeta)|^{p-2}=h'^{p-2}(\zeta)\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}.$$

Hence we have

$$\begin{aligned} Lh(\zeta)=&-\left(h'^{p-1}\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\right)_{x_1}\\ &+\sum_{m=2}^{n}\left(h'^{p-1}\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}F_{x_m}\right)_{x_m}\\ &+g(\mathbf{x},h,\nabla h)+f(\mathbf{x})=-(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\\ &+h'^{p-1}\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\triangle F(\tilde{x})\\ &+(p-2)h'^{p-1}\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-4}{2}}\sum_{m=2}^{n}\left(F_{x_m}\sum_{s=2}^{n}F_{x_s}F_{x_sx_m}\right)\\ &-(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\sum_{m=2}^{n}F_{x_m}^2+g(\mathbf{x},h,\nabla h)+f(\mathbf{x}).\end{aligned}\tag{1.7}$$

Denote by $D^2F(\tilde{x})$ the matrix of second derivatives of $F(\tilde{x})$. Then we can represent the term

$$\sum_{m=2}^{n}\left(F_{x_m}\sum_{s=2}^{n}F_{x_s}F_{x_sx_m}\right)$$

in the following way

$$\sum_{m=2}^{n}\left(F_{x_m}\sum_{s=2}^{n}F_{x_s}F_{x_sx_m}\right)=\nabla F(\tilde{x})D^2F(\tilde{x})(\nabla F(\tilde{x}))^T.\tag{1.8}$$

Thus one can easily see that (1.8) as a bilinear quadratic form is positively defined due to the fact that the matrix $D^2F(\tilde{x})$ as a matrix of second derivatives of concave $F(\tilde{x})$ is positively defined. Moreover, due to the convexity of the domain Ω we have $\triangle F(\tilde{x})\geq 0$. So from (1.7) we obtain the following inequality

$$Lh(\zeta)\geq -(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}-$$

$$(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\sum_{m=2}^{n}F_{x_m}^2+g(\mathbf{x},h,\nabla h)+f(\mathbf{x})\tag{1.9}$$

$$-(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p-2}{2}}\left(1+|\nabla F(\tilde{x})|^2\right)+g(\mathbf{x},h,\nabla h)+f(\mathbf{x})=$$

$$-(p-1)h'^{p-2}h''\left(1+|\nabla F(\tilde{x})|^2\right)^{\frac{p}{2}}+g(\mathbf{x},h,\nabla h)+f(\mathbf{x})\geq$$
$$-(p-1)h'^{p-2}h''+g(\mathbf{x},h,\nabla h)+f(\mathbf{x}).$$

From (1.2) we have that $h''=-C_1h'-C_0$. Therefore (1.9) implies

$$\begin{aligned}Lh(\zeta)&\geq(p-1)C_1h'^{p-1}+(p-1)C_0h'^{p-2}+g(\mathbf{x},h,\nabla h)+f(\mathbf{x})\\&\geq(p-1)C_1h'^{p-1}+(p-1)C_0h'^{p-2}+g(\mathbf{x},0,\nabla h)+f(\mathbf{x}),\end{aligned}$$

where the last inequality follows from the fact that $h\geq 0$ and (0.4). Using the inequality $h'\geq C_0$ we immediately conclude that

$$Lh(\zeta)\geq(p-1)C_0^{p-1}(C_1+1)+g(\mathbf{x},0,\nabla h)+f(\mathbf{x}).$$

Suppose now that $g(x,0,\nabla h)\neq 0$. From (1.1) and (0.8) we obtain that

$$Lh(\zeta)\geq(p-1)C_0^{p-1}(C_1+1)+g(\mathbf{x},0,h'(\zeta),-h'(\zeta)F_{x_2},...,-h'(\zeta)F_{x_n})+f(\mathbf{x})\geq 0.$$

If $g(x,0,\nabla h)=0$ then then due to (1.1), we have

$$(p-1)C_0^{p-1}(C_1+1)\geq f_0$$

and

$$Lh(\zeta)\geq(p-1)C_1h'^{p-1}+(p-1)C_0^{p-1}+f(\mathbf{x})\geq 0.$$

Define $h^*(\mathbf{x})$ by

$$h^*(\mathbf{x})=\begin{cases}h(G(\tilde{x})-x_1), & \text{for}\quad \mathrm{x}_1\geq\frac{\mathrm{F}(\tilde{\mathrm{x}})+\mathrm{G}(\tilde{\mathrm{x}})}{2}\\ h(x_1-F(\tilde{x})), & \text{for}\quad \mathrm{x}_1<\frac{\mathrm{F}(\tilde{\mathrm{x}})+\mathrm{G}(\tilde{\mathrm{x}})}{2}.\end{cases}$$

Function $h^*(\mathbf{x})$ is the viscosity supersolution of (0.1) and $h^*(\mathbf{x})=0$ on $\partial\Omega$. In fact, each of the functions $h(G(\tilde{x})-x_1)$, $h(x_1-F(\tilde{x}))$ are classical supersolutions of (0.1). So the function $h^*(\mathbf{x})$ is also a classical supersolution of (0.1) in the domains

$$\{\mathbf{x}:\mathbf{x}\in\Omega,x_1>\frac{F(\tilde{x})+G(\tilde{x})}{2}\}\ \text{ and }\ \{\mathbf{x}:\mathbf{x}\in\Omega,x_1<\frac{F(\tilde{x})+G(\tilde{x})}{2}\}.$$

On the line $x_1=\frac{F(\tilde{x})+G(\tilde{x})}{2}$ the function $h^*(\mathbf{x})$ is only continuous. From the definition of a viscosity supersolution and due to the continuity of the elliptic operator (0.1) $h^*(\mathbf{x})$ will be a viscosity supersolution of (0.1), (0.2). This fact follows easily from $h'\geq C_0$, which implies that there does not exist any $\mathbb{C}^2$ function whose graph touches the graph of h^* from below at a point belonging to the set $\{x\in\Omega:x_1=\frac{F(\tilde{x})+G(\tilde{x})}{2}\}$.

Now let us show that function $h_*(\mathbf{x})\equiv -h^*(\mathbf{x})$ is a subsolution of (0.1), (0.2). Obviously $h_*(\mathbf{x})=0$ on $\partial\Omega$. For $h_1(\xi)\equiv -h(\xi)=-h(G(\tilde{x})-x_1)$ we have

$$L(h_1(\xi))=L(-h(G(\tilde{x})-x_1))\equiv-\mathrm{div}(|\nabla(-h)|^{p-2}\nabla(-h))+g(\mathbf{x},-h,\nabla(-h))+f(\mathbf{x})$$

$$\mathrm{div}(|\nabla h|^{p-2}\nabla h)+g(\mathbf{x},-h,\nabla(-h))+f(\mathbf{x})\leq \tag{1.10}$$
$$\mathrm{div}(|\nabla h|^{p-2}\nabla h)+g(\mathbf{x},0,\nabla(-h))+f_0,$$

where the last inequality is the consequence of the fact that $-h \leq 0$. From (1.6) we have that

$$Lh(\xi) \geq (p-1)C_0^{p-1}(C_1+1) + g(\mathbf{x}, 0, \nabla h) + f(\mathbf{x}) \geq 0. \tag{1.11}$$

Thus (0.1) and (1.11) imply

$$\operatorname{div}(|\nabla h|^{p-2}\nabla h) \leq -(p-1)C_0^{p-1}(C_1+1). \tag{1.12}$$

Finally, from (1.10), (1.12) we obtain that

$$\begin{aligned} L(h_1(\xi)) \leq& \operatorname{div}(|\nabla h|^{p-2}\nabla h) + g(\mathbf{x}, 0, \nabla(-h)) + f_0 \\ \leq& -(p-1)C_0^{p-1}(C_1+1) + g(\mathbf{x}, 0, \nabla(-h)) + f_0. \end{aligned}$$

Now using (1.1) and (0.7) we conclude that $L(h_1(\xi)) \leq 0$. Similarly, using (1.1) and (0.9), one can prove that for $h_1(\zeta) \equiv -h(x_1 - F(\tilde{x}))$ the inequality $L(h_1(\zeta)) \leq 0$ holds. So due to the continuity of the elliptic operator (0.1) the function

$$h_*(\mathbf{x}) = \begin{cases} -h(G(\tilde{x}) - x_1), & \text{for} \quad \mathrm{x}_1 \geq \frac{\mathrm{F}(\tilde{\mathrm{x}})+\mathrm{G}(\tilde{\mathrm{x}})}{2} \\ -h(x_1 - F(\tilde{x})), & \text{for} \quad \mathrm{x}_1 < \frac{\mathrm{F}(\tilde{\mathrm{x}})+\mathrm{G}(\tilde{\mathrm{x}})}{2}. \end{cases}$$

is a viscosity subsolution of (0.1), (0.2).

Thus we have constructed a supersolution $h^*(\mathbf{x})$ and a subsolution $h_*(\mathbf{x})$ of (0.1), (0.2). Since the comparison principle holds we may invoke the Ishii-Perron method to obtain the existence and uniqueness of the viscosity solution of (0.1), (0.2).

Finally let us give several examples.

Example 1. Obviously if $g(\mathbf{x}, u, \mathbf{q}) + f(\mathbf{x})$ is a continuous function that satisfies conditions (0.4), (0.5) and

$$g(\mathbf{x}, 0, \mathbf{q}) \equiv 0,$$

then conditions of the Theorem are fulfilled.

Example 2. If $g = a|u_{x_1}|^{p-1} + \phi(u)$, where a is a constant, then conditions (0.6) - (0.9) are fulfilled with

$$C_1 = \frac{1}{p-1}\left(a + \frac{|\phi(0)|}{C_0^{p-1}}\right).$$

In fact,

$$|g(\mathbf{x}, 0, |\rho|)| \leq |a||\rho|^{p-1} + |\phi(0)| \leq C_1(p-1)\rho^{p-1}.$$

Hence, conditions of the Theorem are fulfilled if $f(\mathbf{x})$, $\phi(u)$ are continuous functions on $\overline{\Omega}$ and $[-\max h^*, \max h^*]$ respectively and ϕ is strictly increasing.

Example 3. Suppose now that $g = g(\mathbf{x}, u)$. In that case (0.6)–(0.9) are fulfilled with

$$C_1 = \frac{\max_\Omega |g(\mathbf{x}, 0)|}{(p-1)C_0^{p-1}}.$$

Thus conditions of the Theorem are fulfilled if $g(\mathbf{x}, u)$, $f(\mathbf{x})$ are continuous functions on $\overline{\Omega} \times [-\max h^*, \max h^*]$ and $\overline{\Omega}$ respectively and g is strictly increasing with respect to u.

Here existence and uniqueness would also follow from the observation that this is an Euler equation of a strictly convex functional. Therefore a weak solution exists, and weak solutions are often viscosity solutions according to [13].

Example 4. Consider for simplicity the two dimensional case. Suppose that Ω is a domain in $\mathbb{R}^2$ with $\partial\Omega$ consisting of two parabolas, $x_1 = G(x_2) = 1 - x_2^2$ for $x_1 \geq 0$ and $x_1 = F(x_2) = x_2^2 - 1$ for $x_1 \leq 0$. If

$$g = a|\nabla u|^{p-1} + \phi(u),$$

then conditions (0.6)–(0.9) are fulfilled with

$$C_1 = \frac{1}{p-1}\Big(ak_0^{(p-1)/2} + \frac{|\phi(0)|}{C_0^{p-1}}\Big),$$

where $k_0 = \max\{1 + \max G_{x_2}^2, 1 + \max F_{x_2}^2\} = 5$. In fact,

$$|g(\mathbf{x}, 0, |\rho|, |\rho G_{x_2}|)| \leq |a|\rho^{p-1}(1 + G_{x_2}^2)^{(p-1)/2} + |\phi(0)| \leq C_1(p-1)\rho^{p-1}.$$

$$|g(\mathbf{x}, 0, |\rho|, |\rho F_{x_2}|)| \leq |a||\rho|^{p-1}(1 + F_{x_2}^2)^{(p-1)/2} + |\phi(0)| \leq C_1(p-1)\rho^{p-1}.$$

Thus we conclude that conditions of the Theorem are fulfilled if f, ϕ are continuous functions on $\overline{\Omega}$ and $[-\max h^*, \max h^*]$ respectively and ϕ is strictly increasing.

References

[1] L. Caffarelli, M. Crandall, M. Kocan and A. Święch, On viscosity solutions of fully nonlinear equations with measurable ingredients. Comm. Pure Appl. Math. **49**, 365–397 (1996).

[2] M. G. Crandall, Quadratic forms, semidifferentials and viscosity solutions of fully nonlinear elliptic equations. Ann. Inst. H. Poincaré. Anal. Non Linéaire. **6**, 419–435 (1989).

[3] M. G. Crandall and P. L. Lions, Viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc. **277**, 1–42 (1983).

[4] M. G. Crandall and H. Ishii, The maximum principle for semicontinuous functions. Differential Integral. Equations **3**, 1001–1014 (1990).

[5] M. G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. **27**, 1–67 (1992).

[6] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's. Comm. Pure Appl. Math. **42**, 14–45 (1989).

[7] H. Ishii, P. L. Lions, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differential Equations **83**, 26–78 (1990).

[8] H. Ishii and P. Souganidis, Generalized motion of noncompact hypersurfaces with velocity having arbitrary growth on the curvature tensor. Tôhoku Math. J. **47**, 227–250 (1995).

[9] B. Kawohl and N. Kutev, Comparison principle and Lipschitz regularity for viscosity solutions of some classes of nonlinear partial differential equations. Funkcial. Ekvac **43**, 241–253 (2000).

[10] R. Jensen, The maximum principle for viscosity solutions of fully nonlinear second-order partial differential equations. Arch. Rat. Mech. Anal. **101**, 1–27 (1988).

[11] R. Jensen, Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations. Indiana Univ. Math. J. **38**, 629–687 (1989).

[12] R. Jensen, P. L. Lions and P. E. Souganidis, A uniqueness result for viscosity solutions of second-order fully nonlinear partial differential equations. Proc. Amer. Math. Soc. **102**, 975–978 (1988).

[13] P. Juutinen, P. Lindqvist and J. J. Manfredi, On the equivalence of the viscosity solutions and weak solutions for a quasilinear equation. SIAM J. Math. Anal. vol. **33**(N3), 699–717 (2001).

[14] M. Ohnuma and K. Sato, Singular degenerate parabolic equations with applications to the p-Laplace diffusion equation. Comm. Partial Diff. Eq. **22**(3,4), 381–411 (1997).

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Received: 18 January 2006