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Ultraparabolic equations and unsteady heat transfer

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Abstract. The present paper is concerned with the boundary value problem for the equation $\theta_t + \mathbf{u} \cdot \nabla\theta = \kappa\theta_{yy} + f$. Existence and uniqueness of the generalized solution are proved.

1. Introduction

We consider equation

$$\theta_t + \mathbf{u} \cdot \nabla\theta = \kappa\theta_{yy} + f \quad \text{in} \quad Q_{T,X} \tag{0.1}$$

coupled with initial-boundary conditions

$$\theta(0, x, y) = \theta_0(x, y), \quad \theta(t, 0, y) = \theta(t, x, \pm R) = 0. \tag{0.2}$$

Here $\mathbf{u} = \mathbf{u}(t, x, y) = (u(t, x, y), v(t, x, y))$, $\nabla\theta = (\theta_x, \theta_y)$, $\mathbf{u} \cdot \nabla\theta = u\theta_x + v\theta_y$, $f = f(t, x, y, \theta, \theta_y)$,

$$\begin{aligned} &Q_{T,X} = \{(t, x, y) : 0 < t < T, 0 < x < X, |y| < R\}, \\ &\quad u > 0 \ \text{ for } \ |y| < R, \ \ u(t, x, \pm R) \geq 0, \ \ \kappa > 0. \end{aligned} \tag{0.3}$$

Equation (0.1) belongs to the class of ultraparabolic equations. Such equations describe nonstationary transport (of matter, impulse, temperature) processes where in some direction the effect of diffusion is negligible as compared to convection. Ultraparabolic equations were first introduced by A. N. Kolmogorov [3], in the probability treatment of certain diffusion processes. There recently appeared a large number of publications concerning ultraparabolic equations (see [1], [5] and the references there) but only few of them are devoted to boundary value problems. In the case when the coefficient $u = u(t, x)$ and $f = f(t, x, y)$ the solvability of problems (0.1), (0.2) in Hölder spaces follows from [13]. If $u = b_1x + b_2y$, where b_1, b_2 – const then the existence of generalized solution follows from

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[6], [10]. Weak solution of problem (0.1), (0.2) for $f = f(t, x, y)$ was obtained in [11]. The approach proposed in [13] can not be applied in the case when u depends on y or in the quasilinear case, the method used in [6], [10] cannot be extended to arbitrary $u(t, x, y)$ and finally the method developed in [11] cannot be extended to quasilinear case.

Our interest in ultraparabolic equations is motivated by the following. Consider unidirectional flow in a tube with radius R, where x is the axis of the tube (see [2, Sections 4.2, 4.3]). The component of the velocity in the y direction is zero ($v \equiv 0$) and the component in the x direction is a function of time t and spatial variable $y : u = u(t, y)$. The simplest case is Poiseuille flow: $u = u(|y|)$ where $u(\pm R) = 0$ and $u(|y|) > 0$ for $|y| < R$. The unsteady convection-diffusion equation takes the form

$$\theta_t + u\theta_x = \kappa \, \Delta \, \theta + f,$$

where θ is temperature, the positive constant κ is the heat conductivity coefficient and f is a source. It is known (see [7, Section 35]) that if the Péclet number

$$Pe = RePr$$

(here Re is the Reynolds number and Pr is the Prandtl's number) is large with respect to 1, then the convective transfer of heat in the x direction essentially exceeds the molecular transfer (diffusion) and we can reject the term θ_{xx} in the equation. If we treat θ as a concentration of the admixture, the term θ_{xx} can be neglected when the diffusion coefficient κ is small compared with $\tilde{a}R$, where $\tilde{a}$ is the average with respect to y of the velocity in x direction (see [7, Section 21]).

The advantage of such approach is that we need only one boundary condition with respect to x, i.e., it is sufficient to perform the measurement in the x direction only once (at the beginning of the tube $x = 0$).

In order to prove the existence we regularize the original problem by the following one:

$$\theta^\varepsilon_t + u(t, x, y)\theta^\varepsilon_x + v(t, x, y)\theta^\varepsilon_y = \varepsilon\theta^\varepsilon_{xx} + \kappa\theta^\varepsilon_{yy} + f(t, x, y, \theta^\varepsilon, \theta^\varepsilon_y), \tag{0.1)*}$$

$$\theta^\varepsilon(0, x, y) = \theta_0(x, y), \quad \theta^\varepsilon(t, 0, y) = \theta^\varepsilon(t, x, \pm R) = \theta^\varepsilon_x(t, X, y) = 0, \tag{0.2)*}$$

$\varepsilon > 0 -$ const. We obtain apriori estimates independent of ε and pass to the limit $\varepsilon \to 0$ in the integral identity

$$\int_{Q_{T,X}} [\theta^\varepsilon_t + u(t, x, y)\theta^\varepsilon_x + v(t, x, y)\theta^\varepsilon_y - \kappa\theta^\varepsilon_{yy} - f(t, x, y, \theta^\varepsilon, \theta^\varepsilon_y)]\phi dt dx dy$$

$$= -\varepsilon \int_{Q_{T,X}} \theta^\varepsilon_x \phi_x dt dx dy.$$

The key step here is the proof of the boundary gradient estimates independent of ε. In order to pass to the limit in the nonlinear term $f(t, x, y, \theta^\varepsilon, \theta^\varepsilon_y)$ we use the compactness lemma (see Section 3). The derivative with respect to x of the obtained limit function θ has no trace on the boundary $x = X$ and hence "ignores" the condition $\theta_x(t, X, y) = 0$.

Let us make several assumptions on the smoothness of the data and on the structure of nonlinearity. Assume that

$$\operatorname{sign} \theta f(t, x, y, \theta, 0) \le C_0 u(t, x, y) \tag{0.4}$$

for $(t, x, y) \in \bar{Q}_{T,X}$ and any θ. Suppose that

$$|f(t, x, y, \theta, p)| \le \kappa \psi(|p|) \tag{0.5_1}$$

for $(t, x, y) \in Q_{T,X}$, $|\theta| \le M$ and any p, where $M > 0$ is some constant, $\psi(\rho) > 0$ is a smooth function such that

$$\int^{+\infty} \frac{\rho d\rho}{\psi(\rho)} = +\infty. \tag{0.5_2}$$

Assume that function $f(t, x, y, \theta, p)$ for $(t, x, y) \in \bar{Q}_{T,X}$, $\theta \in [-M, M]$ and $q^2 + p^2 > L$ satisfies the following restriction

$$f_\theta + \frac{q f_x + p f_y}{q^2 + p^2} \le C_f, \tag{0.6}$$

where $L > 0$ and $C_f > 0$ are some constants. Concerning the smoothness of the data we suppose that

$$\begin{aligned} &\theta_0 \in C^1([0, X] \times [-R, R]), \quad \theta_0(x, \pm R) = \theta_0(0, y) = 0, \\ &u, v \in C^1(\bar{Q}_{T,X}) \quad f \in C^1(\bar{Q}_{T,X} \times [-M, M] \times \mathbf{R}). \end{aligned} \tag{0.7}$$

DEFINITION. We say that Hölder continuous function $\theta(t, x, y)$ is a generalized solution of problem (0.1), (0.2) if

i) θ satisfies equation (0.1) almost everywhere, initial and boundary conditions are admitted in the classical sense;
ii) $\theta_x, \theta_y \in L_\infty(Q_{T,X})$, $\theta_t, \theta_{yy} \in L_2(Q_{T,X})$.

THEOREM. *Suppose that conditions* (0.3)–(0.7) *are fulfilled. Then there exists a unique generalized solution of problem* (0.1)*,* (0.2).

In order to simplify the notation below we will omit the superscript ε.

§1. A priori Estimates of θ, θ_x and θ_y

In this section we will obtain a priori estimates of θ, θ_x and θ_y in L_∞ norm for the solution of regularized problem (0.1)*, (0.2)*.

We say that θ is a classical solution of (0.1)*, (0.2)* if $\theta \in C^{1;2}_{t;x,y}(Q_{T,X}) \cap C^{0;1}_{t;x,y}(\bar{Q}_{T,X})$. Here $C^{1;2}_{t;x,y}(Q_{T,X})$ is the set of functions having the first derivative with respect to t and the second derivatives with respect to x, y continuous in $Q_{T,X}$.

LEMMA 1.1. *Let* $\theta(t, x, y)$ *be a classical solution of problem* (0.1)*, (0.2)*, *assume that condition* (0.4) *is fulfilled. Then*

$$|\theta(t, x, y)| \leq C_1 x \leq M = C_1 X \ \ in \ \ \bar{Q}_{T,X},$$

where $C_1 = \max\{C_0, \sup|\theta_{0x}|\}$.

Proof. Let

$$L_0\theta \equiv \theta_t - \varepsilon\theta_{xx} - \kappa\theta_{yy}.$$

Introduce the following function

$$h_0(x) \equiv (C_1 + \epsilon)x,$$

where $\epsilon > 0$ − constant. Obviously $L_0 h_0 = 0$ and for the function $\theta(t, x, y) - h_0(x)$ we have

$$L_0(\theta - h_0) = f(t, x, y, \theta, \theta_y) - u(t, x, y)\theta_x - v(t, x, y)\theta_y.$$

Denote by Γ the parabolic boundary of $Q_{T,X}$, i.e., $\Gamma \equiv \partial Q_{T,X} \setminus \{t = T\}$. If $\theta - h_0$ attains its positive maximum at the point $N \in \bar{Q}_{T,X} \setminus \Gamma$ then at this point $\theta - h_0 > 0$, $(\theta - h_0)_x = 0$, $(\theta - h_0)_y = 0$, i.e., $\theta > 0$, $\theta_x = h_0' = C_1 + \epsilon$, $\theta_y = 0$ and hence due to (0.4)

$$L_0(\theta - h_0)|_N = f(N, \theta(N), 0) - u(N)(C_1 + \epsilon) < 0.$$

This contradicts the assumption that $\theta(t, x, y) - h_0(x)$ attains positive maximum at N.

Consider the parabolic boundary Γ. We have $\theta_0(x, y) - h_0(x) = (\theta_0(x, y) - \theta_0(0, y)) - (h_0(x) - h_0(0)) = (\theta_{0x} - h_0')x \leq 0$; $\theta(t, 0, y) - h_0(0) = 0$; for $y = \pm R$ we have $\theta - h_0 = -h_0 \leq 0$. Let us show that the function $\theta - h_0$ cannot attain maximum when $x = X$. In fact, suppose that $\theta - h_0$ attains maximum at the point (t_0, X, y_0). Due to (0.2)* we have $(\theta(t_0, X, y_0) - h_0(X))_x = -h_0'(X) < 0$, which is impossible. Consequently we conclude that $\theta(t, x, y) \leq h_0(x)$ in $\bar{Q}_{T,X}$.

Now consider function $\theta(t, x, y) + h_0(x)$. If $\theta + h_0$ attains its negative minimum at the point $N_1 \in \bar{Q}_{T,X} \setminus \Gamma$ then at this point we have $\theta + h_0 < 0$, $(\theta + h_0)_x = 0$, $(\theta + h_0)_y = 0$, i.e., $\theta < 0$, $\theta_x = -h_0' = -(C_1 + \epsilon)$, $\theta_y = 0$ and hence due to (0.4)

$$L_0(\theta + h_0)|_{N_1} = f(N_1, \theta(N_1), 0) + u(N_1)(C_1 + \epsilon) > 0.$$

This contradicts the fact that $\theta + h_0$ attains negative minimum in the interior of the domain $Q_{T,X}$.

Consider the parabolic boundary Γ. We have $\theta_0(x, y) + h_0(x) = (\theta_{0x} + h_0')x \geq 0$; $\theta(t, 0, y) + h_0(0) = 0$ and for $y = \pm R$ we have $\theta + h_0 = h_0 \geq 0$. Let us show that the function $\theta + h_0$ cannot attain minimum when $x = X$. In fact, suppose that $\theta + h_0$ attains minimum at the point (t_1, X, y_1). Due to (0.2)* we have $(\theta(t_1, X, y_1) + h_0(X))_x = h_0'(X) > 0$, which is impossible. Consequently we conclude that $\theta(t, x, y) \geq h_0(x)$ in $\bar{Q}_{T,X}$.

Passing to the limit when $\epsilon \to 0$ we fulfil the proof. □

REMARK. Suppose that $\theta_0 \equiv 0$ and $f = f(t, x, y)$. In this case condition (0.4) takes the form $|f(t, x, y)| \leq C_0 u(t, x, y)$. From Lemma 1.1 it follows that for the fixed source f choosing the velocity u sufficiently big we can make $|\theta|$ arbitrary small. The explanation of this phenomenon is simple. Cooling is caused by the transfer of the substance with zero temperature from the boundary $x = 0$ along the tube and depends on the velocity of propagation of this substance.

Now introduce the functions $h_1(y)$ and $h_2(y) \equiv h_1(-y)$ by the following

$$h_1'' = -(\psi_1(|h'|) + \epsilon), \quad h_1(-R) = 0, \quad h_1(-R + y_0) = M,$$

y_0 will be defined below. Here

$$\psi_1(\rho) \equiv \psi(\rho) + \frac{\max|v(t, x, y)|}{\kappa}\rho.$$

Represent the solution of this problem in parametric form using the standard substitution $q = h_1'$:

$$h_1(q) = \int_q^{q_1} \frac{\rho d\rho}{\psi_1(\rho) + \epsilon}, \quad y(q) = \int_q^{q_1} \frac{d\rho}{\psi_1(\rho) + \epsilon} - R,$$

select $q_1 > q_2 \geq K$ such that

$$\int_{q_0}^{q_1} \frac{\rho d\rho}{\psi_1(\rho) + \epsilon} = M,$$

this is possible due to (0.5_2). Put

$$y_0 = \int_{q_0}^{q_1} \frac{d\rho}{\psi_1(\rho) + \epsilon}.$$

Obviously $h_1'(y) \geq K$.

LEMMA 1.2. *Let* $\theta(t,x,y)$ *be a classical solution of problem* (0.1)*, (0.2)* *and assume that conditions* (0.5) *are fulfilled, then*

$$|\theta(t,x,y)| \le h_1(y) \;\; in \;\; \bar{D}_1, \;\; |\theta(t,x,y)| \le h_2(y) \;\; in \;\; \bar{D}_2,$$

where

$$D_1 = \{(t,x,y) : 0 < t < T, 0 < x < X, y \in (-R, -R + y_0) \cap (-R, R)\},$$

$$D_2 = \{(t,x,y) : 0 < t < T, 0 < x < X, y \in (R - y_0, R) \cap (-R, R)\}.$$

Proof. Consider function $\theta(t,x,y) - h_1(y)$. Define operator L_1

$$L_1 \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} - \varepsilon\frac{\partial^2}{\partial x^2} - \kappa\frac{\partial^2}{\partial y^2}.$$

Obviously

$$\begin{aligned} L_1(\theta - h_1) &= f(t,x,y,\theta,\theta_y) - v(t,x,y)h_1' + \kappa h_1'' \\ &= f(t,x,y,\theta,\theta_y) - vh_1' - \kappa\psi(|h_1'|) - \max|v|h_1' - \kappa\epsilon \\ &\le f(t,x,y,\theta,\theta_y) - \kappa\psi(|h_1'|) - \kappa\epsilon. \end{aligned}$$

Suppose that $\theta - h_1$ attains positive maximum at $N \in \bar{D}_1 \setminus \Gamma_1$ where Γ_1 is the parabolic boundary of D_1. Then $(\theta - h_1)_y(N) = 0$ and hence at this point $\theta_y = h_1'$. From (0.5_1) it follows that

$$L_1(\theta - h_1)|_N \le f(t,x,y,\theta,h_1') - \kappa\psi(|h_1'|) - \kappa\epsilon|_N < 0$$

which is impossible at the point where $\theta - h_1$ attains positive maximum. Consider the parabolic boundary of D_1. For $y = -R$ we have $\theta - h_1 = -h_1(-R) = 0$. For $t = 0$ we have $\theta_0(x,y) \le h_1$, since $h_1'(y) \ge K$ and for $x = 0$ obviously $\theta(t,0,y) = 0 \le h_1$. If $y_0 < 2R$ then for $y = -R + y_0$, we have $\theta - h_1 \le M - h_1(-R + y_0) = 0$; if $y_0 \ge 2R$ then we compare θ and h_1 on the boundary $y = R$ where we have $\theta - h_1 = -h_1(R) \le 0$.

Let us show now that $\theta - h_1$ cannot attain positive maximum at a point (t_*, X, y_*) where $0 < t_* < T$, $y_* \in (-R, -R + y_0) \cap (-R, R)$. Assume that at the point (t_*, X, y_*) the function $\theta - h_1$ attains positive maximum. Introduce function

$$\Theta(t,x,y) \equiv \theta(t,x,y) - h_1(y) - \theta_* + \epsilon\chi(x)$$

where $\theta_* = \theta(t_*, X, y_*) - h_1(y_*)$, $\chi(x) = e^{-\alpha x^2} - e^{-\alpha X^2}$, positive constants ϵ, α will be selected below. Consider Θ in

$$D_1^{x_0} = \{(t,x,y) : 0 < t < T, x_0 < x < X, y \in (-R, -R + y_0) \cap (-R, R)\},$$

where $0 < x_0 < X$. Select $\alpha = (2x_0^2)^{-1}$, we have

$$\begin{aligned} L_1(\Theta) &= f(t, x, y, \theta, \theta_y) - v h_1' - \kappa\psi_1(|h_1'|) - \kappa\epsilon + \epsilon L_1(\chi) \\ &\le f(t, x, y, \theta, \theta_y) - \kappa\psi(|h_1'|) - \kappa\epsilon, \end{aligned}$$

because in $D_1^{x_0}$

$$L_1(\chi) = -2\alpha e^{-\alpha x^2}(xu + \varepsilon(2\alpha x^2 - 1)) \le 0.$$

Function Θ cannot attain positive maximum at $N_* \in \bar{D}_1^{x_0} \setminus \Gamma^{x_0}$ where Γ^{x_0} is the parabolic boundary of $D_1^{x_0}$. In fact, at N_* we have $\Theta_y = 0$, i.e., $\theta_y = h_1'$ and

$$L_1(\Theta)|_{N_*} \le f(t, x, y, \theta, h_1') - \kappa\psi(|h_1'|) - \kappa\epsilon|_{N_*} < 0$$

which is impossible. Consider Θ on Γ^{x_0}. For $x = X$ we have

$$\Theta(t, X, y) = \theta(t, X, y) - h_1(y) - \theta_* \le 0.$$

Selecting ϵ sufficiently small we obtain

$$\Theta(t, x_0, y) = \theta(t, x_0, y) - h_1(y) - \theta_* + \epsilon\chi(x_0) \le 0.$$

Here we use the inequality $\theta(t, x_0, y) - h_1(y) < \theta_*$. This inequality follows from the fact that $\theta - h_1$ cannot attain positive maximum in $\bar{D}_1 \setminus \Gamma_1$ and from the assumption that $\theta - h_1$ attains positive maximum at (t_*, X, y_*). Besides for sufficiently small ϵ we have

$$\begin{aligned} &\Theta(t, x, -R) = -\theta_* + \epsilon\chi(x) \le 0, \\ &\Theta(t, x, -R + y_0) = \theta(t, x, -R + y_0) - M - \theta_* + \epsilon\chi(x) \le 0. \end{aligned}$$

and

$$\Theta(0, x, y) = \theta_0(x, y) - h_1(y) - \theta_* + \epsilon\chi(x) \le 0.$$

Thus $\Theta \le 0$ in $\bar{D}_1^{x_0}$, i.e.,

$$\theta(t_*, x, y_*) - h_1(y_*) - (\theta(t_*, X, y_*) - h_1(y_*)) \le -\epsilon\chi(x) \text{ in } \bar{D}_1^{x_0}$$

and hence $(\theta(t_*, x, y_*) - h_1(y_*))_x|_{x=X} = 2\alpha\epsilon X e^{-\alpha X^2} > 0$. This contradicts the boundary condition $(\theta(t, X, y) - h_1(y))_x = \theta_x(t, X, y) = 0$. Hence $\theta - h_1$ cannot attain positive maximum at $x = X$, $0 < t < T$, $|y| < R$.

Due to the facts that $\theta - h_1 \le 0$ on the rest parts of the parabolic boundary of D_1 and that $\theta - h_1$ cannot attain positive maximum at $D_1 \setminus \Gamma_1$ we conclude

$$\theta - h_1 \le 0 \text{ in } \bar{D}_1.$$

Now consider function $\theta(t,x,y)+h_1(y)$. Obviously for $y=-R$, for $t=0$, for $x=0$ and for $y=-R+y_0$ (or $y=R$) we have $\theta+h_1\geq 0$. Suppose that $\theta+h_1$ attains negative minimum at $N_1\in\bar{D}_1\backslash\Gamma_1$. Then at this point we have $(\theta+h_1)_y=0$ and hence $\theta_y=-h_1'$. From (0.5) we obtain

$$L_1(\theta+h_1)|_{N_1}=f(t,x,y,\theta,-h_1')+vh_1'+\kappa\psi_1(|h'|)+\kappa\epsilon|_{N_1}$$

$$\geq f(t,x,y,\theta,-h_1')+\kappa\psi(|h'|)+\kappa\epsilon|_{N_1}>0.$$

Similarly to the previous case we can show that if negative minimum is obtained when $x=X$ then at this point $(\theta+h_1)_x<0$, this contradicts the boundary condition $(\theta+h_1)_x|_{x=X}=0$. As a consequence we conclude that

$$\theta+h_1\geq 0 \text{ in } \bar{D}_1$$

and finally

$$|\theta|\leq h_1 \text{ in } \bar{D}_1.$$

Analogously we can establish the estimate $|\theta(t,x,y)|\leq h_2(y)$ in D_2. Lemma is proved. □

REMARK. The estimates of Lemma 1.1 and Lemma 1.2 imply the following boundary gradient estimates:

$$|\theta_x||_{x=0}\leq C_1,\quad |\theta_y||_{y=\pm R}\leq h_1'(-R)=-h_2'(R).$$

As it was mentioned in the introduction these estimates play key role in all procedure of obtaining apriori estimates independent of ε. The global estimates of θ_x,θ_y in L_∞ norm (Lemma 1.3) as well as the estimates θ_t,θ_{yy} in L_2 norm (Lemmas 2.1, 2.2) can be obtained now by a standard method. For completeness we will give a brief proof.

LEMMA 1.3. *Suppose that conditions* (0.4), (0.5) *are fulfilled, then for any classical solution of problem* (0.1)*, (0.2)* *the following estimate holds*

$$\sup_{\bar{Q}_{T,X}}(\theta_x^2+\theta_y^2)\leq C_2,$$

where constant C_2 *depends only on* L, T, $\sup|u_x|$, $\sup|v_y|$, $\sup|v_x+u_y|$, $\sup|\nabla\theta_0|^2$ *and* $C_1, h_1'(-R), C_f$.

Proof. Differentiate equation (0.1)* with respect to x and multiply by $2\theta_x$, then differentiate equation (0.1)* with respect to y and multiply by $2\theta_y$. Summing up for $w\equiv\theta_x^2+\theta_y^2$

we obtain

$$\begin{aligned}&\kappa w_{yy} + \varepsilon w_{xx} - w_t - u w_x - (v - f_p) w_y \\&\quad + 2 f_\theta w + 2\theta_x f_x + 2\theta_y f_y - 2(v_x + u_y)\theta_x\theta_y - 2u_x\theta_x^2 - 2v_y\theta_y^2 \\&\quad - 2\kappa(\theta_{yy}^2 + \theta_{xy}^2) - 2\varepsilon(\theta_{xy}^2 + \theta_{xx}^2) = 0.\end{aligned}$$

Thus we have

$$\begin{aligned}&\kappa w_{yy} + \varepsilon w_{xx} - w_t - u w_x - (v - f_p) w_y \\&\quad + 2\left(f_\theta + \frac{\theta_x f_x + \theta_y f_y}{\theta_x^2 + \theta_y^2} + a_0 \right) w \geq 0,\end{aligned}$$

where $a_0 = \max\{\sup|u_x|, \sup|v_y|\} + \frac{1}{2}\sup|v_x + u_y|$. From the lemma on the normal derivative taking into account that $w_x(t, X, y) = 0$, we conclude that w cannot attain maximum at $x = X$ for $|y| < R$, $0 < t < T$. Hence

$$w \leq \max\{L, e^{KT} \sup_\Gamma |w|\},$$

where $2(C_f + a_0) \leq K$ and Γ is parabolic boundary of $Q_{T,X}$. For more details see, for example, [8] Chapter 11. Lemma is proved. □

§2. A priori Estimates of θ_{yy} and θ_t

In this section we will obtain a priori estimates of θ_{yy} and θ_t in L_2 norm for the solution of problem (0.1)*, (0.2)*. Denote

$$U = \sup u(t, x, y), \quad F = \sup|f(t, x, y, \theta, \theta_y) - v(t, x, y)\theta_x|,$$

here supremum is taken over the set $\bar{Q}_{T,X} \times [-M, M] \times [-\sqrt{C_2}, \sqrt{C_2}]^2$.

LEMMA 2.1. *For any classical solution of problem* (0.1)*, (0.2)* *such that* $\theta_{tx}, \theta_{xyy} \in L_2(Q_{T,X})$ *we have*

$$\int_{Q_{T,X}} \theta_{yy}^2 \, dt\,dx\,dy \leq C_3,$$

where the constant C_3 *depends only on* U, C_2, F, $\|\theta_{0y}\|_{L_2((0,X)\times(-R,R))}$, T, X *and* κ^{-1}.

Proof. Multiply equation (0.1)* by θ_{yy} and integrate by part with respect to y and x and then integrate with respect to t, we obtain:

$$\frac{1}{2}\int_0^X\int_{-R}^R \theta_y^2 dydx + \int_{Q_{T,X}} \theta_{xy}^2 dtdxdy + \int_{Q_{T,X}} \kappa\theta_{yy}^2 dtdxdy$$
$$= \frac{1}{2}\int_0^X\int_{-R}^R \theta_{0y}^2 dxdy + \int_{Q_{T,X}} u\theta_x\theta_{yy} dtdxdy - \int_{Q_{T,X}} (f - v\theta_y)\theta_{yy} dtdxdy.$$

From the Cauchy inequality and the Young inequality $|c||d| \le \frac{1}{p}\delta^p|c|^p + \frac{1}{q}\delta^{-q}|d|^q$ with $p = q = 2$ and $\delta^2 = 2/3$ we have

$$\left|\int_{Q_{T,X}} u\theta_x\theta_{yy} dtdxdy\right| \le \left(\int_{Q_{T,X}} \kappa^{-1}u^2\theta_x^2 dtdxdy\right)^{1/2}\left(\int_{Q_{T,X}} \kappa\theta_{yy}^2 dtdxdy\right)^{1/2}$$
$$\le \frac{1}{3}\int_{Q_{T,X}} \kappa\theta_{yy}^2 dtdxdy + \frac{3}{4}\int_{Q_{T,X}} \kappa^{-1}u^2\theta_x^2 dtdxdy.$$

Similarly

$$\left|\int_{Q_{T,X}} (f - v\theta_y)\theta_{yy} dtdxdy\right| \le \frac{1}{3}\int_{Q_{T,X}} \kappa\theta_{yy}^2 dtdxdy$$
$$+\frac{3}{4}\int_{Q_{T,X}} \kappa^{-1}(f - v\theta_y)^2 dtdxdy.$$

Finally

$$\frac{1}{2}\int_0^X\int_{-R}^R \theta_y^2 dydx + \frac{1}{3}\int_{Q_{T,X}} \theta_{yy}^2 dtdxdy$$
$$\le \frac{1}{2}\int_0^X\int_{-R}^R \theta_{0y}^2 dydx + \frac{3}{4}\int_{Q_{T,X}} \kappa^{-1}(u^2\theta_x^2 + (f - v\theta_y)^2) dtdxdy$$
$$\le \frac{1}{2}\int_0^X\int_{-R}^R \theta_{0y}^2 dydx + \frac{3}{2\kappa}(U^2C_2^2 + F^2)RTX.$$

$\square$

Lemma is proved.

LEMMA 2.2. *For any classical solution of problem* (0.1)*, (0.2)* *such that* $\theta_{tx}, \theta_{ty} \in L_2(Q_{T,X})$ *we have*

$$\int_{Q_{T,X}} \theta_t^2 dtdxdy \le C_4,$$

where the constant C_4 *depends only on* U, F, C_2, C_3 *and* $\varepsilon\|\theta_{0x}\|_{L_2((0,X)\times(-R,R))}$.

Proof. Multiply equation (0.1)* by θ_t and integrate over $Q_{T,X}$ to obtain:

$$\int_{Q_{T,X}} \theta_t^2 dtdxdy + \frac{1}{2}\varepsilon \int_0^X\int_{-R}^R \theta_x^2(T,x,y)dydx$$
$$= \int_{Q_{T,X}} (\kappa\theta_{yy}\theta_t - u\theta_x\theta_t + (f - v\theta_y)\theta_t)dtdxdy + \frac{1}{2}\varepsilon \int_0^X\int_{-R}^R \theta_{0x}^2 dydx.$$

Now applying the Cauchy and the Young inequalities we obtain

$$\int_{Q_{T,X}} (\kappa\theta_{yy}\theta_t - u\theta_x\theta_t + (f - v\theta_y)\theta_t)dtdxdy$$
$$\le \frac{3}{4}\int_{Q_{T,X}} \theta_t^2 dtdxdy + \int_{Q_{T,X}} (\kappa^2\theta_{yy}^2 + u^2\theta_x^2 + (f - v\theta_y)^2)dtdxdy.$$

Hence

$$\int_{Q_{T,X}} \theta_t^2 dtdxdy \le 4\int_{Q_{T,X}} (\kappa^2\theta_{yy}^2 + u^2\theta_x^2 + (f - v\theta_y)^2)dtdxdy$$
$$+ 2\varepsilon \int_0^X\int_{-R}^R \theta_{0x}^2 dydx.$$

□

Lemma is proved.

§3. Existence and uniqueness

For the sake of simplicity we require $\theta_{0x}(X, y) = 0$. Assumptions (0.4)–(0.7) guarantee the existence of the solution of problem (0.1)*, (0.2)* belonging to $C^{1;2}_{t;x,y}(Q_{T,X}) \cap C^{0;1}_{t;x,y}(\bar{Q}_{T,X})$ such that $\theta^\varepsilon_{tx}, \theta^\varepsilon_{ty}, \theta^\varepsilon_{xyy}, \theta^\varepsilon_{yxx} \in L_2(Q_{T,X})$ (see [4], [8]). From the estimates obtained in previous sections it follows that we can find a subsequence ε_k such that

$$\theta^{\varepsilon_k} \to \theta,\ \ \theta^{\varepsilon_k}_x \to \theta_x,\ \ \theta^{\varepsilon_k}_y \to \theta_y,\ \ \text{*weakly in } L_\infty(Q_{T,X}),$$

when $k \to +\infty$ ($\varepsilon_k \to 0$). Moreover (see [12]),

$$|\theta^\varepsilon(t_1, x, y) - \theta^\varepsilon(t_2, x, y)| \le C|t_1 - t_2|^{1/2},$$

where the constant C depends only on $\sup|\nabla\theta^\varepsilon|$ and does not depend on ε^{-1}. So we have $\theta^{\varepsilon_k} \to \theta$ uniformly (in C^0 norm) and the limit function θ is Hölder continuous. From the obtained a priori estimates of θ^ε_t and θ^ε_{yy} in L_2 norm we conclude:

$$\theta^{\varepsilon_k}_t \to \theta_t,\ \ \theta^{\varepsilon_k}_{yy} \to \theta_{yy},\ \ \text{weakly in } L_2(Q_{T,X}) \text{ when } k \to +\infty.$$

Recall the following compactness lemma (see [9], Chapter 1, Section 5). Let B_0, B, B_1 are Banach spaces, B_0, B_1 are reflexive, $B_0 \subset B \subset B_1$, and the embedding $B_0 \subset B_1$ is compact. Let

$$\mathcal{W} \equiv \{g | g \in L_2(0, T; B_0), g_t \in L_2(0, T; B_1)\},$$

define the norm in $\mathcal{W}$ by

$$\|g\|_{\mathcal{W}} \equiv \|g\|_{L_2(0,T;B_0)} + \|g_t\|_{L_2(0,T;B_1)}.$$

LEMMA. *The embedding $\mathcal{W} \subset L_2(0, T; B)$ is compact.*

Let us take

$$B_0 \equiv W^{1;2}_{x;y}((0, X) \times (-R, R)), B \equiv W^{0;1}_{x;y}((0, X) \times (-R, R)),$$
$$B_1 \equiv L_2((0, X) \times (-R, R)).$$

Thus we have that

$$\theta^{\varepsilon_k}_y \to \theta_y \text{ in } L_2(Q_{T,X}) \text{ norm when } k \to +\infty.$$

Consider the following integral identity

$$\int_{Q_{T,X}} [\theta^{\varepsilon_k}_t + u(t, x, y)\theta^{\varepsilon_k}_x + v(t, x, y)\theta^{\varepsilon_k}_y - \kappa\theta^{\varepsilon_k}_{yy} - f(t, x, y, \theta^{\varepsilon_k}, \theta^{\varepsilon_k}_y)]\phi dt dx dy$$
$$= -\varepsilon_k \int_{Q_{T,X}} \theta^{\varepsilon_k}_x \phi_x dt dx dy,$$

where $\phi, \phi_x \in L_2(Q_{T,X})$. Passing to the limit when $k \to +\infty$ we obtain the required solution.

Uniqueness can be proved by the standard method based on Gronwall's inequality. In fact if we suppose that there exists two solution θ_1 and θ_2 then the function $\theta \equiv \theta_1 - \theta_2$ satisfies the integral identity

$$\int_{Q_{t,X}} (\theta_t + u\theta_x)\phi d\tau dx dy$$
$$= \int_{Q_{t,X}} (\kappa\theta_{yy} - v\theta_y + (f(t, x, y, \theta_1, \theta_{1y}) - f(t, x, y, \theta_2, \theta_{2y}))\phi d\tau dx dy.$$

Taking $\phi = \theta$ for

$$\omega(t) \equiv \frac{1}{2}\int_0^X \int_{-R}^R \theta^2 dx dy$$

we obtain

$$\omega(t) \leq \int_0^t \Phi(\tau)\omega(\tau) dt,$$

where $\Phi(\tau) \geq 0$ belongs to $L_1(0, T)$. From Gronwall's inequality it follows that $\omega(t) \equiv 0$ and hence $\theta_1(t, x, y) \equiv \theta_2(t, x, y)$.

REFERENCES

[1] AKHMETOV, D., LAVRENTIEV, M. JR. and SPIGLER, R., *Singular perturbations for parabolic equations with unbounded coefficients leading to ultraparabolic equations*, Differential Integral Equations *17*(1–2) (2004), 99–118.

[2] BATCHELOR, G. K., *An introduction to fluid dynamics*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1999, 615 pp.

[3] KOLMOGOROV, A. N., *Zufallige Bewegungen*, Ann. of Math. *35*(2) (1934), 116–117.

[4] KONDRATIEV, V. A. and OLEINIK, O. A., *Boundary value problems for partial differential equations in non-smooth domains (Russian)*, Uspekhi Mat. Nauk, *38*(2) (1983), 3–76.

[5] LANCONELLI, E., PASCUCCI, A. and POLIDORO, S., *Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance*, Nonlinear problems in math. physics and related topics, II, 243–265, Int. Math. Ser. (N.Y.), 2, Kluwer/Plenum, New York, 2002.

[6] LASCIALFARI, F. and MORBIDELLI, D., *A boundary value problem for a class of quasilinear ultraparabolic equations*, Comm. Partial Differential Equations *23*(5–6) (1998), 847–868.

[7] LEVICH, V. G., *Physico-chemical Hydrodynamics*, 1962 Prentice Hall.

[8] LIEBERMAN, G., *Second Order Parabolic Equations*, World Scientific Publishing Co., Inc. River Edge, NJ, 1996.

[9] LIONS, J.-L., *Quelques methodes de resolution des problemes aux limites non lineaires*, [French] Dunod; Gauthier-Villars, Paris 1969, 554pp.

[10] MANFREDINI, M., *The Dirichlet problem for a class of ultraparabolic equations*, Adv. Differential Equations *2*(5) (1997), 831–866.

[11] PYATKOV, S. G., *Solvability of boundary value problems for an ultraparabolic equation*, Nonclassical equations of mixed type [Russian], 182–197, Akad. Nauk SSSR Sibirsk. Otdel., Inst. Mat. Novosibirsk, 1990.

[12] TERSENOV, AL. S. and TERSENOV, AR. S., *The Cauchy problem for a class of quasilinear parabolic equations*, Ann. Mat. Pura Appl. *182*(3) (2003), 325–336.

[13] TERSENOV, S. A., *Basic boundary value problems for one ultraparabolic equation*, Siberian Math. J. *42*(6) (2001), 1173–1190.

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