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# On the Bernstein-Nagumo's condition in the theory of nonlinear parabolic equations

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**Abstract.** The present paper is concerned with the Bernstein-Nagumo's condition for nonlinear and quasilinear parabolic equations. We show that Bernstein-Nagumo's condition can be substituted by a less restrictive one. We prove the existence theorems for the boundary value problems and discuss several examples of gradient blow-up in order to show the optimality of the new condition.

## Introduction

In the present paper we consider the nonlinear equation

$$u_t = F(t, x, u, u_x, u_{xx}) \quad \text{in } Q_T = (-l, l) \times (0, T), \tag{0.1}$$

coupled with one of the boundary conditions

$$u_x(t, -l) = u_x(t, l) = 0, \tag{0.2}$$

$$u_x + \sigma_1(t, x, u)|_{x=-l} = u_x + \sigma_2(t, x, u)|_{x=l} = 0, \tag{0.3}$$

$$u(t, -l) = u(t, l) = 0, \tag{0.4}$$

and the initial condition

$$u(0, x) = u_0(x). \tag{0.5}$$

We suppose that $F(t, x, u, p, r)$ is continuously differentiable with respect to $r$ function satisfying the parabolicity condition, i.e.

$$F_r(t, x, u, p, r) > 0 \quad \text{for } (t, x, u, p, r) \in \bar{Q}_T \times [-M, M] \times \mathbb{R}^2. \tag{0.6}$$

Let us write equation (0.1) in the following form:

$$u_t = F_r(t, x, u, u_x, \lambda u_{xx})u_{xx} + F(t, x, u, u_x, 0), \quad \lambda \in [0, 1], \tag{0.7}$$

using the mean value theorem. The well known Bernstein-Nagumo's condition [5], [6], [25] (see also [7], [19]–[21], [23], [24], [26]) for equation (0.7) appears as

$$\frac{|F(t,x,u,p,0)|}{F_r(t,x,u,p,r)} \leqq \phi(|p|) \quad \text{for } (t,x,u,p,r) \in \bar{Q}_T \times [-M,M] \times \mathbb{R}^2, \tag{0.8}$$

where $\phi(\rho)$ is nondecreasing positive function such that

$$\int^{+\infty} \frac{\rho\, d\rho}{\phi(\rho)} = +\infty.$$

Condition (0.8) guarantees the global a priori estimate of the gradient of the bounded solution. This a priori estimate plays a key role in proving the existence theorems. There are examples showing that a violation of the Bernstein-Nagumo's condition can imply the gradient blow up on the boundary as well as at interior points of the domain (see [1], [2], [9]–[12], [22], [27], [32]), i.e. there exists a $t^*$ such that $|u_x(t,x_0)| \to +\infty$ when $t \to t^*$ at least for some $x_0 \in [-l,l]$ while the solution itself remains bounded.

The goal of this paper is to substitute condition (0.8) by a less restrictive one which allows an arbitrary growth of $F(t,x,u,p,0)$ with respect to $p$. Of course the examples mentioned above do not satisfy this new condition. There are several papers on this subject for the quasilinear equations [28]–[31]. We want to mention here that the results of the present paper are new for quasilinear equations as well.

Let us formulate the main results. Suppose that the right hand side of equation (0.7) can be represented in the following way:

$$F(t,x,u,p,0) = f_1(t,x,u,p) + f_2(t,x,u,p), \tag{0.9}$$

where function $f_2$ satisfies the next restrictions

$$f_2(t,y,u_1,p) - f_2(t,x,u_2,p) \geqq 0, \tag{0.10}$$

$$f_2(t,x,u_1,-p) - f_2(t,y,u_2,-p) \geqq 0 \tag{0.11}$$

for $t \in [0,T]$, $-l \leqq y < x \leqq l$, $-M \leqq u_1 < u_2 \leqq M$, $p \geqq 0$. For the Dirichlet boundary value problem we additionally suppose that

$$uf_2(t,x,u,p) \leqq 0, \tag{0.12}$$

for $(t,x) \in \bar{Q}_T$, $|u| \leqq M$ and arbitrary $p$.

For the function $f_1$ we assume that

$$|f_1(t,x,u,p)| \leqq F_r(t,x,u,p,r)\psi(|p|) \tag{0.13}$$

for $(t,x) \in \bar{Q}_T$, $|u| \leqq M$ and arbitrary $(p,r)$, where $\psi(\rho) \in \mathbb{C}^1(0,+\infty)$ is a nondecreasing nonnegative function. Suppose that $\psi$ satisfies the next condition: there exist $p_0$ and $p_1$ such that $0 < p_0 < p_1 < +\infty$ and

$$\int_{p_0}^{p_1} \frac{\rho\,d\rho}{\psi(\rho)} \geqq \operatorname{osc}(u) \equiv \max u - \min u. \tag{0.14}$$

Introduce function $h(\tau)$ as a solution of the following problem:

$$h'' + \psi(|h'|) = 0, \quad h(0) = 0, \quad h(\tau_0) = \operatorname{osc}(u),$$

where $\tau_0$ will be specified below. Represent the solution of the equation $h'' + \psi(|h'|) = 0$ in parametrical form (using the standard substitution $h'(\tau) = q(h)$, $\frac{dq}{d\tau} = q\frac{dq}{dh}$):

$$h(q) = \int_{q}^{q_1} \frac{\rho\,d\rho}{\psi(\rho)}, \quad \tau(q) = \int_{q}^{q_1} \frac{d\rho}{\psi(\rho)}.$$

The parameter $q$ varies in the interval $[q_0, q_1]$ and we select $q_0, q_1$ such that $0 < q_0 < q_1 < +\infty$, $h(q_0) = \operatorname{osc}(u)$ (this is possible due to (0.14)). Put $\tau_0 \equiv \tau(q_0)$. Suppose that the initial function satisfies the assumption

$$|u_0(x) - u_0(y)| \leqq h(|x - y|). \tag{0.15}$$

If conditions (0.14), (0.15) as well as conditions (0.10), (0.11) are fulfilled then the gradient of a bounded solution of problem (0.1), (0.2), (0.5) is bounded by a constant depending only on $\psi, \operatorname{osc}(u)$. In the case of problem (0.1), (0.3), (0.5) we need additional assumption on $p_0$ in terms of functions $\sigma_i$ (see Lemma 2). For problem (0.1), (0.4), (0.5) assumptions (0.10)–(0.12), (0.14), (0.15) guarantee the gradient estimate of a bounded solution of this problem depending only on $\psi$ and $\operatorname{osc}(u)$.

Note that (0.15) is a smallness restriction on the $\operatorname{osc}(u_0)$. If $u_0(x)$ is an arbitrary Lipschitz continuous function, then in condition (0.14) and in the parametrical representation of $h(\tau)$ we must take $p_0 = K$ and $q_0 = K$ respectively, where $|u_0(x) - u_0(y)| \leqq K|x - y|$ and we do not need restriction (0.15) (strictly speaking this restriction will be automatically fulfilled (see proof of Lemma 1)). Thus for an arbitrary Lipschitz continuous function $u_0(x)$ conditions (0.14), (0.15) are equivalent to the following one: there exists $p_1 > K$ such that

$$\int_{K}^{p_1} \frac{\rho\,d\rho}{\psi(\rho)} \geqq \operatorname{osc}(u). \tag{0.16}$$

The global gradient a priori estimate

$$|u_x(t, x)| \leqq C$$

holds with $C$ depending only on $\psi, K, \operatorname{osc}(u)$.

If $f_1(t, x, u, p) \equiv 0$, then for a solution of problems (0.1), (0.2), (0.5) and (0.1), (0.4), (0.5) we have:

$$\max_{Q_T} |u_x(t, x)| \leqq K. \tag{0.17}$$

If $f_1(t,x,u,p)$ is an arbitrary function satisfying the Bernstein-Nagumo's condition (0.8), then for a classical solution of problems (0.1), (0.2), (0.5) and (0.1), (0.4), (0.5) holds:

$$\max_{Q_T} |u_x(t,x)| \leqq C, \tag{0.18}$$

where the constant $C$ depends only on $\phi, K$ and $\operatorname{osc}(u)$.

Note that if $f_1$ satisfies (0.8) then for an arbitrary Lipschitz continuous function $u_0(x)$ the gradient estimate (0.18), (0.17) holds and condition (0.16) is automatically fulfilled for any $K$.

One can easily construct function $f_2$ satisfying conditions (0.10), (0.11) or (0.10)–(0.12) and having an arbitrary growth with respect to $p$ (see Section 2). Note that conditions on $f_2$ are independent of the principal part of equation (0.7) (i.e. of $F_r$).

In the case when $f_2 \equiv 0$ condition (0.16) first appears in [15], [16] for quasilinear equation and in [3] for fully nonlinear equation.

In order to prove the existence theorems additional assumptions are needed. Specifically we require conditions (2.1)–(2.3) (see Section 2) to be fulfilled. Conditions (2.1), (2.2) guarantee the a priori estimate of $\max|u|$, while (2.3) is an assumption on the smoothness of the function $F(t,x,u,p,r)$. Now let us formulate the existence theorems.

**Theorem 1.** *Suppose that conditions* (0.6), (0.9)–(0.13), (0.16), (2.1), (2.3) *hold and* $u_0(x) \in \mathbb{C}^{1+\beta}([-l,l])$, *where* $u_0(\pm l) = 0$. *Then for any* $T \in (0,\infty)$ *there exists a solution of problem* (0.1), (0.4), (0.5) *which belongs to* $\mathbb{C}^{1+\gamma/2,2+\gamma}_{t,x}(Q_T) \cap \mathbb{C}^{\gamma,1+\gamma}_{t,x}(\bar{Q}_T)$ *for some* $\gamma \in (0,1)$.

**Theorem 2.** *Suppose that conditions* (0.6), (0.9)–(0.11), (0.13), (0.16), (2.1), (2.3) *hold and* $u_0(x) \in \mathbb{C}^{1+\beta}([-l,l])$. *In addition assume that* $u_0'(-l) = u_0'(l) = 0$. *Then for any* $T \in (0,\infty)$ *there exists a solution of problem* (0.1), (0.2), (0.5) *which belongs to*

$$\mathbb{C}^{1+\gamma/2,2+\gamma}_{t,x}(Q_T) \cap \mathbb{C}^{\gamma,1+\gamma}_{t,x}(\bar{Q}_T) \quad \textit{for some } \gamma \in (0,1).$$

**Theorem 3.** *Suppose that conditions* (0.6), (0.9)–(0.11), (0.13), (0.16), (2.1)–(2.3) *hold and* $u_0(x) \in \mathbb{C}^{1+\beta}([-l,l])$. *In addition assume that*

$$u_0'(-l) + \sigma_1\big(0,-l,u(0,-l)\big) = u_0'(l) + \sigma_2\big(0,l,u(0,l)\big) = 0.$$

*Then for any* $T \in (0,\infty)$ *there exists a solution of problem* (0.1), (0.3), (0.5) *which belongs to* $\mathbb{C}^{1+\gamma/2,2+\gamma}_{t,x}(Q_T) \cap \mathbb{C}^{\gamma,1+\gamma}_{t,x}(\bar{Q}_T)$ *for some* $\gamma \in (0,1)$.

**Theorem 4.** *Suppose that conditions* (0.6), (0.9)–(0.11), (0.13), (0.16), (2.1), (2.3) *hold and* $u_0(x) \in \mathbb{C}^{1+\beta}([-l,l])$ *for any* $|l| < +\infty$ *and vanishes with its first derivative when* $|x| \to \infty$. *Then for any* $T \in (0,\infty)$ *there exists a solution of problem* (0.1), (0.5) *which belongs to* $\mathbb{C}^{1+\gamma/2,2+\gamma}_{t,x}(\Pi_T) \cap \mathbb{C}^{\gamma,1+\gamma}_{t,x}(\bar{\Pi}_T)$ *for some* $\beta \in (0,1)$. *Here* $\Pi_T = (0,T) \times \mathbb{R}$.

Note that in Theorems 1–4 instead of condition (0.16) one can require the fulfilment of conditions (0.14), (0.15).

In Section 1 we obtain the a priori estimate of the gradient of a bounded solution. In Section 2, based on the estimate of the gradient, we prove the existence theorems. In Section 3 we apply the results of Section 1 to quasilinear equations as well as to Hamilton-Jacobi equations. We give also several examples (including the equation of curvature evolution for capillary surfaces and KPZ equation) where we show that in some sense the results of the present paper are optimal.

Before we pass to the next section let us recall that historically restrictions on the growth of the right side with respect to the gradient for nonlinear equations was first formulated in [5]. Specifically in [5] the boundary value problems for the ordinary differential equation $y''(x) = g(x, y(x), y'(x))$ were considered. Under the assumption

$$|g(x, y, p)| \leqq A(x, y)p^2 + B(x, y) \tag{0.19}$$

the gradient $(y'(x))$ estimate was obtained, here $A$ and $B$ are bounded for bounded $x$ and $y$. Based on this estimate and on the estimate of $|y(x)|$ the existence of a classical solution was proved. In [25] condition (0.19) was substituted by the less restrictive one

$$|g(x, y, p)| \leqq \phi(|p|), \quad \int^{+\infty} \frac{\rho\, d\rho}{\phi(\rho)} = +\infty. \tag{0.20}$$

In [14] (0.20) was improved, it was shown that instead of (0.20) $g$ must satisfy:

$$|g(x, y, p)| \leqq \phi(|p|), \quad \int_0^{+\infty} \frac{\rho\, d\rho}{\phi(\rho)} > 2\max|y|. \tag{0.21}$$

Condition (0.16) is actually the analogue of condition (0.21) for parabolic partial differential equations.

## §1. The gradient estimates

In this section we will obtain gradient a priori estimates of classical solutions for boundary value problems for equation (0.1). Recall that a classical solution is a function belonging to $C^{1,2}_{t,x}(Q_T) \cap C^{0,1}_{t,x}(\bar{Q}_T)$ in the case of problem (0.1), (0.2), (0.5) or (0.1), (0.3), (0.5) and to $C^{1,2}_{t,x}(Q_T) \cap C^0(\bar{Q}_T)$ for problem (0.1), (0.4), (0.5). We use here the Kruzhkov's idea of introducing a new spatial variable [19], [20].

Assume that the function $F(t, x, u, p, r)$ is defined for $(t, x) \in \bar{Q}_T$, $u \in [-M, M]$ and arbitrary $(p, r)$ and is bounded on every compact set in $\bar{Q}_T \times [-M, M] \times \mathbb{R}^2$. Suppose that $F$ is differentiable with respect to $r$ and satisfies (0.6). Consider problem (0.1), (0.2), (0.5).

**Lemma 1.** *Let $u(t, x)$ be a classical solution of problem* (0.1), (0.2), (0.5). *Suppose that conditions* (0.6), (0.9)–(0.11), (0.13)–(0.15) *are fulfilled. Then in $\bar{Q}_T$ the inequality*

$$|u_x(t, x)| \leqq C_1$$

*holds, where the constant $C_1$ depends only on* osc$(u)$ *and $\psi$.*

*Proof.* Consider equation (0.1) in the form (0.7) at two different points $(t,x)$ and $(t,y)$:

$$u_t = F_r(t,x,u,u_x,\lambda u_{xx})u_{xx} + F(t,x,u,u_x,0), \quad \lambda \in [0,1],\ u = u(t,x), \tag{1.1}$$

$$u_t = F_r(t,y,u,u_y,\mu u_{yy})u_{yy} + F(t,y,u,u_y,0), \quad \mu \in [0,1],\ u = u(t,y). \tag{1.2}$$

Introduce the function $v(t,x,y) = u(t,x) - u(t,y)$. In

$$\Omega = \{(t,x,y) : 0 < t < T, 0 < x - y, |x| < l, |y| < l\}$$

the function $v(t,x,y)$ satisfies the following equation:

$$\begin{aligned} &-v_t + F_r\big(t,x,u(t,x),u_x(t,x),\lambda u_{xx}(t,x)\big)v_{xx} + F_r\big(t,y,u(t,y),u_y(t,y),\mu u_{yy}(t,y)\big)v_{yy} \\ &\quad = F\big(t,y,u(t,y),u_y(t,y),0\big) - F\big(t,x,u(t,x),u_x(t,x),0\big). \end{aligned} \tag{1.3}$$

Put

$$F_r^{(x)} = F_r\big(t,x,u(t,x),v_x,\lambda u_{xx}(t,x)\big), \quad F_r^{(y)} = F_r\big(t,y,u(t,y),-v_y,\mu u_{yy}(t,y)\big),$$

obviously $u_x(t,x) = v_x$, $u_y(t,y) = -v_y$. Define the operator

$$L(v) \equiv -v_t + F_r^{(x)}[v_{xx} + \psi(|v_x|)] + F_r^{(y)}[v_{yy} + \psi(|v_y|)].$$

From (0.9), (0.13) it follows that

$$L(v) \geqq f_2\big(t,y,u(t,y),u_y(t,y)\big) - f_2\big(t,x,u(t,x),u_x(t,x)\big). \tag{1.4}$$

Let the function $h(\tau)$ be a solution of the following ordinary differential equation:

$$h''(\tau) + \psi\big(|h'(\tau)|\big) = 0 \tag{1.5}$$

on the interval $[0,\tau_0]$ and satisfies conditions

$$h(0) = 0, \quad h(\tau_0) = \operatorname{osc}(u), \quad h' > 0 \quad \text{for } \tau \in [0,\tau_0]. \tag{1.6}$$

Represent the solution of (1.5), (1.6) in parametrical form:

$$h(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \quad \tau(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)}.$$

The parameter $q$ varies in the interval $[q_0, q_1]$, where $0 < q_0 < q_1 < +\infty$ and

$$h(q_0) = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} = \operatorname{osc}(u). \tag{1.7}$$

Put

$$\tau_0 \equiv \tau(q_0) = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)}.$$

Consider the function $w(t,x,y) = v(t,x,y) - h(x-y)$ in

$$P = \{(t,x,y) : 0 < t < T, 0 < x-y < \tau_0, |x| < l, |y| < l\}.$$

Due to the fact that $h(\tau)$ satisfies (1.5) we have $L\big(h(x-y)\big) = 0$. Hence, using (1.4) we obtain

$$\begin{aligned}\tilde{L}(w) &\equiv L(v) - L(h) \equiv -w_t + F_r^{(x)}[w_{xx} + \alpha_1 w_x] + F_r^{(y)}[w_{yy} + \alpha_2 w_y] \\ &\geqq f_2\big(t,y,u(t,y),u_y(t,y)\big) - f_2\big(t,x,u(t,x),u_x(t,x)\big).\end{aligned}$$

Where $|\alpha_i| < +\infty$, $i = 1,2$, by virtue of the mean value theorem and of the fact that $\psi$ is a smooth function and $u$ is a classical solution of (0.1), (0.2), (0.5). Let $\tilde{w} = we^{-t}$, then

$$\begin{aligned}\text{(1.8)} \qquad \tilde{L}_1(\tilde{w}) &\equiv -\tilde{w}_t + F_r^{(x)}[\tilde{w}_{xx} + \alpha_1 \tilde{w}_x] + F_r^{(y)}[\tilde{w}_{yy} + \alpha_2 \tilde{w}_y] - \tilde{w} \\ &\geqq e^{-t}\big[f_2\big(t,y,u(t,y),u_y(t,y)\big) - f_2\big(t,x,u(t,x),u_x(t,x)\big)\big].\end{aligned}$$

Denote by $\Gamma$ the parabolic boundary of $P$

$$(\text{i.e. } \Gamma = \partial P \backslash \{(t,x,y) : t = T, 0 < x-y < \tau_0, |x| < l, |y| < l\}).$$

Suppose that the function $\tilde{w}$ attains its positive maximum at some point $(t_1,x_1,y_1) \in \bar{P} \backslash \Gamma$. Obviously it should be $\tilde{L}_1(\tilde{w})|_{(t_1,x_1,y_1)} < 0$. On the other hand, at this point we have

$$-\tilde{w} < 0, \quad \tilde{w}_x = \tilde{w}_y = 0, \quad \tilde{w}_{xx} \leqq 0, \quad \tilde{w}_{yy} \leqq 0, \quad -\tilde{w}_t \leqq 0,$$

i.e.

$$\begin{aligned}\tilde{w}(t_1,x_1,y_1) &= e^{-t}[u(t_1,x_1) - u(t_1,y_1) - h(x_1-y_1)] > 0, \\ \tilde{w}_x(t_1,x_1,y_1) &= e^{-t}[u_x(t_1,x_1) - h'(x_1-y_1)] = 0, \\ \tilde{w}_y(t_1,x_1,y_1) &= e^{-t}[-u_y(t_1,y_1) + h'(x_1-y_1)] = 0\end{aligned}$$

and as a consequence

$$\text{(1.9)} \qquad u(t_1,x_1) > u(t_1,y_1), \quad u_x(t_1,x_1) = u_y(t_1,y_1) = h'(x_1-y_1) > 0.$$

Hence, from (1.8), (1.9), (0.10) it follows that $\tilde{L}_1\big(\tilde{w}(t_1,x_1,y_1)\big) \geqq 0$. From this contradiction we conclude that $\tilde{w}$ cannot attain its positive maximum in $\bar{P} \backslash \Gamma$.

Now let us show that $\tilde{w}|_\Gamma \leqq 0$. Consider two possible cases: $\tau_0 < 2l$ and $\tau_0 \geqq 2l$. First let $\tau_0 < 2l$. For $t = 0$:

$$\tilde{w}(0,x,y) = e^{-t}\big(u_0(x) - u_0(y) - h(x-y)\big) \leqq 0$$

due to (0.15). Obviously $\tilde{w}(t,x,y)|_{x=y} = 0$ and when $x - y = \tau_0$ we have $\tilde{w} = e^{-t}\big(u(t,x) - u(t,y) - h(\tau_0)\big) \leqq 0$ due to (1.6). Denote by

$$Q_1 = \{(t,x) : 0 < t \leqq T, -l < x < -l + \tau_0, y = -l\},$$
$$Q_2 = \{(t,y) : 0 < t \leqq T, l - \tau_0 < y < l, x = l\}.$$

Estimate the normal derivative of $\tilde{w}$ on $Q_1$ and $Q_2$ using boundary conditions (0.2) and the fact that $h' \geqq q_0 > 0$

$$-\tilde{w}_y(t,x,-l) = e^{-t}\big(u_y(t,-l) - h'(x+l)\big) = -e^{-t}h'(x+l) < 0,$$
$$\tilde{w}_x(t,l,y) = e^{-t}\big(u_x(t,l) - h'(l-y)\big) = -e^{-t}h'(l-y) < 0.$$

Thus the function $\tilde{w}(t,x,y)$ cannot attain its positive maximum neither on $Q_1$ nor on $Q_2$ since $-\partial/\partial y$ and $\partial/\partial x$ are here outward normal derivatives with respect to $P$. Consequently, $\tilde{w}|_\Gamma \leqq 0$ and hence $\tilde{w}(t,x,y) \leqq 0$ in $\bar{P}$.

The case when $\tau_0 \geqq 2l$ can be treated similarly. The only difference is the absence of the boundary $x - y = \tau_0$. We put

$$\tilde{Q}_1 = \{(t,x) : 0 < t \leqq T, -l < x \leqq l, y = -l\},$$
$$\tilde{Q}_2 = \{(t,y) : 0 < t \leqq T, -l < y < l, x = l\}$$

(note that the line $x = l,\ y = -l$ belongs to $\tilde{Q}_1$). Consequently, $\tilde{w}|_\Gamma \leqq 0$ and hence $\tilde{w}(t,x,y) \leqq 0$ in $\bar{P}$. It means that

$$u(t,x) - u(t,y) \leqq h(x-y) \quad \text{in } \bar{P}. \tag{1.10}$$

Treating similarly the function $\tilde{v}(t,x,y) = u(t,y) - u(t,x)$ one can easily see that for $\tilde{w}_1(t,x,y) = e^{-t}\big(\tilde{v}(t,x,y) - h(x-y)\big)$ we have

$$\tilde{L}_1(\tilde{w}_1) \geqq e^{-t}\big[f_2\big(t,x,u(t,x),u_x(t,x)\big) - f_2\big(t,y,u(t,y),u_y(t,y)\big)\big] \quad \text{in } P.$$

Suppose that the function $\tilde{w}_1$ attains its positive maximum at $(\tilde{t}_1, \tilde{x}_1, \tilde{y}_1) \in \bar{P}\backslash\Gamma$. On the one hand it should be $\tilde{L}_1(\tilde{w}_1)|_{(\tilde{t}_1,\tilde{x}_1,\tilde{y}_1)} < 0$. On the other hand, we have

$$u(\tilde{t}_1,\tilde{y}_1) > u(\tilde{t}_1,\tilde{x}_1), \quad u_x(\tilde{t}_1,\tilde{x}_1) = u_y(\tilde{t}_1,\tilde{y}_1) = -h'(\tilde{x}_1 - \tilde{y}_1) < 0.$$

Using inequality (0.11) we obtain that $\tilde{L}_1(\tilde{w}_1) \geqq 0$. From this contradiction it follows that $\tilde{w}_1$ cannot attain positive maximum in $\bar{P}\backslash\Gamma$.

Consider $\tilde{w}_1$ on $\Gamma$. One can easily see that all considerations concerning the estimate of the function $\tilde{w}$ on the boundary $\Gamma$ can be done without any changes in estimate of the function $\tilde{w}_1$. Thus we have that

$$u(t,y) - u(t,x) \leqq h(x-y) \quad \text{in } \bar{P}. \tag{1.11}$$

Combining (1.11) with (1.10) we get

$$|u(t,x) - u(t,y)| \leqq h(x-y) \quad \text{in } \bar{P}.$$

In view of the symmetry of the variables $x, y$ in the same manner we examine the case $y > x$. As a result we have that for

$$0 \leqq t \leqq T, \quad |x| \leqq l, \quad |y| \leqq l, \quad 0 < |x-y| \leqq \tau_0$$

the inequality

$$\left|\frac{u(t,x)-u(t,y)}{x-y}\right| \leqq \frac{h(|x-y|)-h(0)}{|x-y|}$$

holds implying

$$|u_x(t,x)| \leqq h'(0) = q_1 = C_1.$$

Lemma is proved.

**Corollary 1.1.** *Suppose that* $u_0$ *is an arbitrary Lipschitz continuous function* $|u_0(x)-u_0(y)| \leqq K|x-y|$ *and suppose that conditions* (0.6), (0.9)–(0.11), (0.13) *and* (0.16) *are fulfilled. Then for any classical solution of problem* (0.1), (0.2), (0.5) *we have*

$$|u_x(t,x)| \leqq \tilde{C}_1,$$

*where* $\tilde{C}_1$ *depends only on* $\operatorname{osc}(u), \psi$, *and* $K$.

The proof of this statement is a slight modification of the proof of Lemma 1. The first difference is in (1.7). Here we select $q_0 = K$ instead of $q_0 > 0$. The second difference is in the proof of the inequality $\tilde{w}(0,x,y) \leqq 0$. Here we have

$$\tilde{w}(0,x,y) = e^{-t}\big((u_0(x)-u_0(y)) - (h(x-y)-h(0))\big) \leqq e^{-t}\big(K(x-y) - h'(\tau^*)(x-y)\big) \leqq 0$$

since $h' = q \geqq q_0 \geqq K$.

Let us pass to problem (0.1), (0.3), (0.5).

**Lemma 2.** *Let* $u(t,x)$ *be a classical solution of* (0.1), (0.3), (0.5) *and all conditions of Lemma* 1 *are fulfilled. Then in* $\bar{Q}_T$ *the inequality*

$$|u_x(t,x)| \leqq C_2$$

*holds, where the constant* $C_2$ *depends only on* $\operatorname{osc}(u), N_1, N_2$ *and* $\psi$, *where* $N_i = \sup|\sigma_i|$ (*the supremum is taken over the set* $[0,T]\times[-M,M]$).

*Proof.* The proof of Lemma 2 differs from the proof of the previous one only in the selection of $q_0$ and in analysing the behaviour of $\tilde{w}(t,x,y)$ on the bounds $Q_1$ $(\tilde{Q}_1)$ and $Q_2$ $(\tilde{Q}_2)$. We select the quantity $q_0$ so that

$$q_0 > \max\{N_1, N_2\}. \tag{1.12}$$

Taking into account (1.12) and boundary conditions (0.3) we obtain that

$$(1.13)\qquad \begin{aligned} -\tilde{w}_y|_{Q_1} &= -e^{-t}\big(-u_y(t,-l)+h'(x+l)\big)\\ &= e^{-t}\big(-\sigma_1 - h'(x+l)\big) \leqq e^{-t}(N_1-q_0)<0,\\ \tilde{w}_x|_{Q_2} &= e^{-t}\big(u_x(t,l)-h'(l-y)\big)\\ &= e^{-t}\big(-\sigma_2 - h'(l-y)\big) \leqq e^{-t}(N_2-q_0)<0; \end{aligned}$$

$$(1.14)\qquad \begin{aligned} -\tilde{w}_{1y}|_{Q_1} &= -e^{-t}\big(u_y(t,-l)+h'(x+l)\big)\\ &= e^{-t}\big(\sigma_1 - h'(x+l)\big) \leqq e^{-t}(N_1-q_0)<0,\\ \tilde{w}_{1x}|_{Q_2} &= e^{-t}\big(-u_x(t,l)-h'(l-y)\big)\\ &= e^{-t}\big(\sigma_2 - h'(l-y)\big) \leqq e^{-t}(N_2-q_0)<0. \end{aligned}$$

Inequalities (1.13), (1.14) imply that neither $\tilde{w}$ nor $\tilde{w}_1$ can attain positive maximum on $Q_1$ $(\tilde{Q}_1)$ and $Q_2$ $(\tilde{Q}_2)$. By using a similar arguments as in Lemma 1 we complete the proof.

**Corollary 1.2.** *Suppose that* $u_0$ *is an arbitrary Lipschitz continuous function* $|u_0(x)-u_0(y)| \leqq K|x-y|$ *and suppose that conditions* (0.6), (0.9)–(0.11), (0.13) *and* (0.16) *are fulfilled. Then for any classical solution of problem* (0.1), (0.3), (0.5) *we have*

$$|u_x(t,x)| \leqq \tilde{C}_2,$$

*where* $\tilde{C}_2$ *depends only on* $\operatorname{osc}(u), \psi, N_1, N_2$ *and* $K$.

In order to prove this statement we follow the proof of Corollary 1.1, where we take $q_0 > \max\{K, N_1, N_2\}$.

Consider now problem (0.1), (0.4), (0.5). In that case we additionally suppose that for $|u| \leqq M$ the function $f_2$ satisfies condition

$$(1.15)\qquad uf(t,x,u,p) \leqq 0 \quad \text{for } x \in [-l, -l+\min\{\tau_0, 2l\}] \cup [l-\min\{\tau_0, 2l\}, l].$$

**Lemma 3.** *Let* $u(t,x)$ *be a classical solution of* (0.1), (0.4), (0.5) *and all conditions of Lemma* 1 *are fulfilled. Suppose in addition that condition* (1.15) *is fulfilled and* $u_0(\pm l) = 0$. *Then in* $\bar{Q}_T$ *the following inequality*

$$|u_x(t,x)| \leqq C_3$$

*holds, where the constant* $C_3$ *depends only on* $\operatorname{osc}(u)$ *and* $\psi$.

*Proof.* The proof of Lemma 3 differs from the proof of Lemma 1 only in analysing the behaviour of $w(t,x,y)$ on $Q_1$ $(\tilde{Q}_1)$ and $Q_2$ $(\tilde{Q}_2)$.

Let us show that $w(t,x,y) \leqq 0$ on $\bar{Q}_2$. When $x = l$, we have

$$w(t,l,y) = -u(t,y) - h(l-y).$$

Define the following linear operator $L_0(u) \equiv -u_t + F_r^{(y)} u_{yy}$, obviously

$$L_0(u) = -f_1(t, y, u, u_y) - f_2(t, y, u, u_y)$$

and

$$L_0\big(h(l - y)\big) = F_r^{(y)} h_{yy} = -F_r^{(y)} \psi(|h'|).$$

For $w_2 \equiv u(t, y) + h(l - y)$ we have

$$L_0(w_2) = -f_1(t, y, u, u_y) - f_2(t, y, u, u_y) - \tilde{F}^{(y)} \psi(|h'|).$$

Let us show that $w_2(t, y) \geqq 0$ on $\bar{Q}_2$. For $\tilde{w}_2 = w_2 e^{-t}$ we have

$$\tilde{L}_0(\tilde{w}_2) \equiv -\tilde{w}_{2t} + F_r^{(y)} \tilde{w}_{2yy} - \tilde{w}_2 = e^{-t}[-f_1(t, y, u, u_y) - f_2(t, y, u, u_y) - F_r^{(y)} \psi(|h'|)].$$

If the function $\tilde{w}_2(t, y)$ attains its negative minimum at some point $(t_1, y_1) \in Q_2$ then at this point it should be $\tilde{L}_0(\tilde{w}_2) > 0$. At the same time due to (0.13) and taking into account that $u_y(t_1, y_1) = h'(l - y_1)$ (because $\tilde{w}_{2y}(t_1, y_1) = 0$) we have

$$-f_1\big(t_1, y_1, u(t_1, y_1), h'\big) - F_r^{(y)} \psi(|h'|) \leqq 0.$$

On the other hand by virtue of the fact that

$$\tilde{w}_2(t_1, y_1) < 0 \quad \text{and hence} \quad u(t_1, y_1) < -h(l - y_1) \leqq 0,$$

using inequality (1.15) we obtain

$$-f_2\big(t_1, y_1, u(t_1, y_1), h'(l - y_1)\big) \leqq 0.$$

As a consequence

$$\tilde{L}_0(\tilde{w}_2)|_{(t_1, y_1)} \leqq 0.$$

From this contradiction we conclude that $\tilde{w}_2$ cannot attain its negative minimum on $Q_2$. Let us show that $\tilde{w}_2 \geqq 0$ on parabolic boundary of $Q_2$. As in Lemma 1 we will consider two cases. First we suppose that $\tau_0 < 2l$. One can easily see that condition (0.15) together with $u_0(\pm l) = 0$ give us

$$|u_0(y)| \leqq h(l - y). \tag{1.16}$$

Due to (1.16) we have

$$\tilde{w}_2(0, y) = e^{-t}[u_0(y) + h(l - y)] \geqq 0.$$

For $y = l - \tau_0$ we have $\tilde{w}_2(t, l - \tau_0) = e^{-t}\big(u(t, l - \tau_0) + h(\tau_0)\big) \geqq 0$ (recall that $h(\tau_0) = \operatorname{osc} u$ and $\operatorname{osc}(u) \geqq \max|u|$ because $u(t, l) = 0$). For $y = l$ we have

$$\tilde{w}_2(t, l) = e^{-t}\big(u(t, l) + h(0)\big) = 0$$

due to (0.4) and (1.6). The case when $\tau_0 > 2l$ can be treated similarly. The only difference which makes the things simpler is the substitution of the boundary $y = l - \tau_0$ by $y = -l$. Hence $w(t, l, y) \leqq 0$ on $\bar{Q}_2$.

Consider the function $w_1 = u(t, y) - u(t, x) - h(x - y)$. For $x = l$ we have $w_1 = u(t, y) - h(l - y)$. Let us show that $w_3 \equiv u(t, y) - h(l - y) \leqq 0$ on $\bar{Q}_2$. For $w_3(t, y)$ we have

$$L_0(w_3) = -f_1(t, y, u, u_y) - f_2(t, y, u, u_y) + F_r^{(y)}\psi(|h'|),$$

and for $\tilde{w}_3 = w_3 e^{-t}$ we have

$$\tilde{L}_0(\tilde{w}_3) \equiv -\tilde{w}_{3t} + F_r^{(y)}\tilde{w}_{3yy} - \tilde{w}_3 = e^{-t}[-f_1(t, y, u, u_y) - f_2(t, y, u, u_y) + F_r^{(y)}\psi(|h'|)].$$

If $\tilde{w}_3(t, y)$ attains its positive maximum at some point $(t_2, y_2) \in Q_2$, then it should be $\tilde{L}_0(\tilde{w}_3)|_{(t_2,y_2)} < 0$. From the other hand

$$u(t_2, y_2) > h(l - y_2) \geqq 0, \quad u_y(t_2, y_2) = -h'(l - y_2)$$

and hence due to (0.13), (1.15) we conclude

$$-f_1\big(t_2, y_2, u(t_2, y_2), -h'(l - y_2)\big) + F_r^{(y)}\psi\big(|h'(l - y_2)|\big) \\ - f_2\big(t_2, y_2, u(t_2, y_2), -h'(l - y_2)\big) \geqq 0.$$

Thus we obtain that $\tilde{L}_0(\tilde{w}_3)|_{(t_2,y_2)} \geqq 0$, which in turn contradicts the assumption that $\tilde{w}_3$ attains its positive maximum. One can easily obtain that $\tilde{w}_3 \leqq 0$ on the parabolic boundary of $Q_2$. Whence it immediately follows that $w_1(t, l, y) \leqq 0$ on $\bar{Q}_2$. Note that here we use the fact that $|u_0(x)| \leqq h(x + l)$, which is the consequence of (0.15) and $u_0(\pm l) = 0$.

Let us show now that $w(t, x, y) \leqq 0$ and $w_1(t, x, y) \leqq 0$ on $\bar{Q}_1$. When $y = -l$ we have

$$w(t, x, -l) = u(t, x) - u(t, -l) - h(x + l) = u(t, x) - h(x + l).$$

$$w_1(t, x, -l) = u(t, -l) - u(t, x) - h(x + l) = -u(t, x) - h(x + l).$$

Using the same arguments when proving that $w_3(t, y) \leqq 0$ on $\bar{Q}_2$ one can obtain that $w_4(t, x) = w(t, x, -l) \leqq 0$ on $\bar{Q}_1$. And finally using the same arguments when proving that $w_2(t, x) \geqq 0$ on $\bar{Q}_2$ one can obtain that $w_5(t, x) = -w_1(t, x, -l) \geqq 0$ on $\bar{Q}_1$. By using similar arguments as in Lemma 1, we complete the proof.

**Corollary 1.3.** *Suppose that $u_0$ is an arbitrary Lipschitz continuous function $|u_0(x) - u_0(y)| \leqq K|x - y|$ and suppose that conditions* (0.6), (0.9)–(0.13) *and* (0.16) *are fulfilled. Then for any classical solution of problem* (0.1), (0.2), (0.5) *we have*

$$|u_x(t, x)| \leqq \tilde{C}_3,$$

*where $\tilde{C}_3$ depends only on* $\operatorname{osc}(u), \psi$, *and $K$.*

The proof is similar to the proof of Corollary 1.1.

**Corollary 1.4.** *Suppose that all conditions of Lemma* 3 *except condition* (1.15) *are fulfilled. If* $\tau_0 \leqq l$ *and the function* $f_2$ *is independent of* $u$, *suppose that* (*instead of* (1.15)) $f_2$ *satisfies conditions*

$$pf_2(t,x,p) \leqq 0 \quad \text{for } x \in [-l, -l+\tau_0], \tag{1.17}$$

$$pf_2(t,x,p) \geqq 0 \quad \text{for } x \in [l-\tau_0, l]. \tag{1.18}$$

*Then in* $\bar{Q}_T$ *the following inequality*

$$|u_x(t,x)| \leqq C_3'$$

*holds, where the constant* $C_3'$ *depends only on* $\operatorname{osc}(u)$ *and* $\psi$.

*Proof.* Consider the function $w = u(t,x) - u(t,y) - h(x-y)$. Following the proof of Lemma 3 we obtain

$$\tilde{L}_0(\tilde{w}_2) \equiv -\tilde{w}_{2t} + F_r^{(y)}\tilde{w}_{2yy} - \tilde{w}_2 = e^{-t}[-f_1(t,y,u,u_y) - F_r^{(y)}\psi(|h'|) - f_2(t,y,u_y)].$$

If the function $\tilde{w}_2(t,y)$ attains its negative minimum at some point $(t_1,y_1) \in Q_2$, then at this point $\tilde{L}_0(\tilde{w}_2) > 0$. On the other hand, by virtue of the fact that

$$u_y(t_1,y_1) = h'(l-y_1) > 0,$$

using inequality (1.18) and condition (0.13) we obtain

$$-f_1\big(t_1,y_1,u,h'(l-y_1)\big) - F_r^{(y)}\psi\big(|h'(l-y_1)|\big) - f_2\big(t_1,y_1,h'(l-y_1)\big) \leqq 0.$$

As a consequence

$$\tilde{L}_0(\tilde{w}_2)|_{(t_1,y_1)} \leqq 0.$$

From this contradiction we conclude that $\tilde{w}_2$ cannot attain its negative minimum on $Q_2$. Similarly to the proof of Lemma 3 we conclude that $w(t,l,y) \leqq 0$ on $\bar{Q}_2$.

Consider the function $w_1 = u(t,y) - u(t,x) - h(x-y)$. Following the proof of Lemma 3 we obtain

$$\tilde{L}_0(\tilde{w}_3) \equiv -\tilde{w}_{3t} + F_r^{(y)}\tilde{w}_{3yy} - \tilde{w}_3 = e^{-t}[-f_1(t,y,u,u_y) + F_r^{(y)}\psi(|h'|) - f_2(t,y,u_y)].$$

If $\tilde{w}_3(t,y)$ attains its positive maximum at some point $(t_2,y_2) \in Q_2$ then at this point $\tilde{L}_1(\tilde{w}_3) < 0$. On the other hand

$$u_y(t_2,y_2) = -h'(l-y_2) < 0,$$

and hence due to (0.13), (1.18)

$$\tilde{L}_0(\tilde{w}_3)|_{(t_2,y_2)} \geqq 0.$$

This contradicts the assumption that $\tilde{w}_3$ attains its positive maximum. Following the proof of Lemma 3 we conclude that $w_1(t,l,y) \leqq 0$ on $\bar{Q}_2$.

Using inequality (1.17) we similarly obtain that $w(t,x,y) \leqq 0$, $w_1(t,x,y) \leqq 0$ on $\bar{Q}_1$ and complete the proof.

**Remark 1.** Note that if the function $f_1$ satisfies the Bernstein-Nagumo's condition (0.8) then we can always select $\tau_0$ to be less or equal to $l$.

In fact, select $p_0 \geqq \operatorname{osc}(u) l^{-1}$ then

$$\tau_0 = \int_{p_0}^{p_1} \frac{d\rho}{\psi(\rho)} \leqq \frac{1}{p_0} \int_{p_0}^{p_1} \frac{\rho\, d\rho}{\psi(\rho)} = \frac{\operatorname{osc}(u)}{p_0} \leqq l.$$

**Remark 2.** Our assumptions on the functions $f_1(t,x,u,p)$ and $f_2(t,x,u,p)$ appearing in Lemmas 1, 2, 3, can be somehow weakened. One can easily see that in order to prove the above mentioned lemmas those assumptions must be fulfilled only for $p$ from $[-p_1, -p_0] \cup [p_0, p_1]$.

Consider the case $F(t,x,u,p,0) = f_2(t,x,u,p)$ where the gradient estimates take a very simple form.

Suppose that all the assumptions of Lemma 1 are fulfilled and in addition $f_1 \equiv 0$. Then for the classical solution of problem (0.1), (0.2), (0.5) we have

$$|u_x(t,x)| \leqq K, \tag{1.19}$$

where $K$ is a Lipschitz constant of $u_0(x)$ (i.e. $|u_0(x) - u_0(y)| \leqq K|x-y|$). In fact, when $f_1 \equiv 0$ we can take $\psi \equiv 0$. So the barrier function is the solution of the equation $h'' = 0$ i.e. $h = K\tau$. It is not difficult to show that $|u(t,x) - u(t,y)| \leqq K|x-y|$ in

$$\bar{P} = \{(t,x,y) : 0 \leqq t \leqq T, 0 \leqq x - y, |x| \leqq l, |y| \leqq l\}$$

and $|u_x| \leqq h'(0) = K$.

Suppose that all the assumptions of Lemma 2 are fulfilled and in addition $f_1 \equiv 0$. In the same way as in the previous example we obtain that for a classical solution of problem (0.1), (0.3), (0.5) we have

$$|u_x(t,x)| \leqq \max\{K, N_1, N_2\}. \tag{1.20}$$

Now suppose that all the assumptions of Lemma 3 are fulfilled and in addition $f_1 \equiv 0$. Analogously to the previous examples we obtain that for a classical solution of problem (0.1), (0.4), (0.5) we have

$$|u_x(t,x)| \leqq K. \tag{1.21}$$

## §2. The existence theorems

In the previous section the a priori estimate of $|u_x|$ was carried out under the condition that $u$ is a bounded solution of boundary value problems. Now let us formulate sufficient conditions that guarantee the boundedness of $u$. Consider problem (0.1), (0.4), (0.5). The fulfilment of the condition

$$uF(t,x,u,0,0) \leqq |u|\Phi(|u|), \quad \int^{+\infty} \frac{dz}{\Phi(z)} = +\infty, \tag{2.1}$$

for $(t,x) \in \bar{Q}_T$ and $|u| \geqq M$, where $M$ is some positive constant and $\Phi(z)$ is a nondecreasing positive function of $z \geqq 0$, guarantees the global a priori estimate of $|u|$ for problem (0.1), (0.4), (0.5) (see [20]). The same condition guarantees the boundedness of the solution of problem (0.1), (0.2), (0.5) ([20]). Consider problem (0.1), (0.3), (0.5). If $F(t,x,u,p,r)$ satisfies condition (2.1) together with

$$u\sigma_1(t,x,u)|_{x=-l} < 0, \quad u\sigma_2(t,x,u)|_{x=l} > 0 \quad \text{for } |u| > M, \tag{2.2}$$

then we have a global a priori estimate of $|u|$ for the given problem. Concerning the oblique derivative problems one can find condition (2.2) in general form in [24], theorem 13.1. Let us formulate the assumption on $F$ which will be necessary for the proof of the existence theorems. Suppose that

$$\begin{aligned} &|F(t_1,x_1,u_1,p_1,r) - F(t_2,x_2,u_2,p_2,r)| \\ &\quad \leqq C(|t_1-t_2|^{1/2} + |x_1-x_2| + |u_1-u_2| + |p_1-p_2|)^{\beta}(b_1+b_2|r|), \end{aligned} \tag{2.3$_1$}$$

$$|F_r(t,x,u,p,r_1) - F_r(t,x,u,p,r_2)| \leqq C|r_1-r_2|^{\beta}, \tag{2.3$_2$}$$

where $C, b_1, b_2$, are positive constants and constant $\beta \in (0,1)$.

In order to prove the existence Theorems 1–4 (see Introduction) we follow the well-known technique based on the a priori estimates of the solution and the fixed point theorem ([24]). Condition (2.1) gives us an a priori estimate of $|u|$ in the case of the first and the second boundary value problems. Condition (2.1) together with (2.2) give us an a priori estimate of $|u|$ in the case of the third boundary value problem. The a priori estimate of $|u_x|$ was obtained in Section 1. The next step is the deriving of a Hölder estimate of $u_x$. Note that the function $w = u_x$ can be treated as a weak solution of the equation

$$w_t = \big(F_r w_x + F(t,x,u,u_x,0)\big)_x,$$

now based on the well known results of Nash-De Giorgi we obtain the Hölder estimate of $w$ and as a consequence of $u_x$ (for more details see [24], theorems 12.2, 12.10). Having all these a priori estimates the dependence of $F(t,x,u,p,r)$ on variables $u$ and $p$ is no longer important and we can consider the equation (0.1) in the form $u_t = F(t,x,u_{xx})$. The existence follows now from [24], Theorem 14.10 (for more details see also Ch. 14, §7). Note that in order to prove the existence theorem for the Cauchy problem we obtain the solution (0.1), (0.5) as a limit of a sequence of solutions of the second boundary problem under an unlimited dilatation of the domain $\bar{Q}_T$ (see for example [31]).

Let us mention here that according to Corollary 1.4, if $f_2$ is independent of $u$ then in Theorem 1 condition (0.12) can be substituted by conditions (1.17), (1.18).

## §3. Application to quasilinear equations and to Hamilton-Jacobi equation

Consider the quasilinear parabolic equation

$$\big(F(t,x,u,p,r) \equiv a(t,x,u,p)r + f(t,x,u,p)\big)$$

$$u_t = a(t,x,u,u_x)u_{xx} + f(t,x,u,u_x) \quad \text{in } Q_T \tag{3.1}$$

coupled with one of boundary conditions (0.2), (0.3), (0.4) and initial condition (0.5).

Condition (0.6) takes the form

$$a(t,x,u,p) > 0 \quad \text{for } (t,x,u,p) \in Q_T \times [-M,M] \times \mathbb{R}.$$

Obviously

$$F(t,x,u,p,0) = f(t,x,u,p) = f_1(t,x,u,p) + f_2(t,x,u,p).$$

Note that the existence theorems for all boundary value problems for equation (3.1) immediately follow from Theorems 1–3.

Let us point out the novelty of the obtained results concerning the quasilinear case. Conditions (0.10), (0.11) on the term $f_2$ first appear in [30] and [31]. In [31] the existence of a classical solution of Cauchy problem (3.1), (0.5) was proved under the assumptions that $f_1$ satisfies condition (0.8) and $f_2$ satisfies (0.10), (0.11). In [30] the existence of a classical solution for the second and the third boundary value problems was obtained under similar restrictions. In the present paper we suppose that $f_1$ satisfies the weaker than (0.8) restriction. Namely $f_1$ satisfies (0.13), (0.14) and $u_0$ satisfies (0.15) (or $f_1$ satisfies (0.13), (0.16)) for arbitrary Lipschiz continuous function.

In [28] the first boundary value problem was considered. The function $f$ was represented as a sum of $f_1$ and $f_2$, where $f_1$ satisfies the classical Bernstein-Nagumo's condition (0.8) and $f_2(t,x,u,p)$ satisfies conditions similar to (0.13), (0.14) and (0.10), (0.11). Both $f_1$ and $f_2$ satisfy condition $|f_i| \leqq a\psi$. The novelty of the present paper is that we do not need restriction $|f_2| \leqq a\psi$ but supplementary we want condition (1.15) to be fulfilled.

In [29] conditions (0.10), (0.11) appear when proving the existence of a radially—symmetric solution of the boundary value problems in multidimensional case for the equation

$$u_t = \varepsilon \triangle u + f_1(t,|\mathbf{x}|,u,|\nabla u|) + f_2(t,|\mathbf{x}|,u,|\nabla u|).$$

The function $f_1$ satisfies here condition (0.8) (as in [30] and [31]).

Now let us discuss several examples in order to show that the results obtained in the present paper are in some sense optimal.

Consider the following problem:

$$u_t = u_{xx} - (x+1/2)(u_x+U)^3 \quad \text{in } (-1/2,1/2) \times (0,T), \tag{3.2}$$

$$u(0,x) = u_0(x) \quad \text{for } |x| < 1/2, \quad u_0(\pm 1/2) = 0,$$

where $U$ is an arbitrary constant, with boundary conditions

$$u(t, \pm 1/2) = 0. \tag{3.3}$$

Let us take $f_1 \equiv 0$, $f_2 = -(x + 1/2)(u_x + U)^3$. In this case ($f_1 \equiv 0$) from Remark 1 it follows that we can select $p_0 \geqq \frac{\operatorname{osc}(u)}{l}$ and hence $\tau_0 \leqq l$. For $p \geqq p_0 \geqq \max\left\{|U|, \frac{\operatorname{osc}(u)}{l}\right\}$ the function $f_2 = -(x + 1/2)(u_x + U)^3$ satisfies conditions (0.10), (0.11) and condition (1.17). From Remarks 1, 2 and Corollary 1.4 it follows that the gradient blow-up cannot occur neither in the interior of the domain nor on the left side ($x = -1/2$) of the parabolic boundary. So we can guarantee that if the gradient blow-up takes place it may occur only on the right side of the parabolic boundary (i.e. $x = 1/2$). In [32] it was shown that for $U = \pi/2$ and for arbitrary initial function $u_0(x) \max|u_x(t, 1/2)| \to +\infty$ as $t$ goes to the proper value $t^*$.

On the other hand, we also can represent the right hand side as a sum of $f_1 = -(x + 1/2)(u_x + U)^3$ and $f_2 = 0$. Then from the results of Section 1 it follows that if

$$\int_0^{+\infty} \frac{\rho\, d\rho}{(\rho + |U|)^3} > \operatorname{osc}(u), \tag{3.4}$$

i.e. $(2|U|)^{-1} > \operatorname{osc}(u) \geqq \max|u|$ (because there exists a point where $u = 0$), then we have the global estimate of the gradient. Condition (3.4) gives us some values of $U$ in order to obtain the global a priori estimate of $u_x$. Note that if $U = \pi/2$, then for arbitrary initial function $u_0(x)$ condition (3.4) fails. In fact, for $U = \pi/2$ we have

$$\int_0^{+\infty} \frac{\rho\, d\rho}{(\rho + |U|)^3} = \frac{1}{\pi}$$

and $\max|u| > \frac{1}{\pi}$ (see [32]).

Consider now the following problem:

$$u_t = u_{xx} + g(u)|u_x|^{m-1} u_x \quad \text{in } (0, T) \times (-1, 1), \tag{3.5}$$

$$u(0, x) = u_0(x), \quad u(t, \pm 1) = A_\pm, \quad u_0(\pm 1) = A_\pm, \tag{3.6}$$

where $A_+, A_-$ are some constants, $m > 2$. Here $ug(u) > 0$ when $|u| > 0$, $g'(u) > 0$ for $0 < |u| < \varepsilon$ and $g \in \mathbb{C}^1(\mathbb{R})$. In [1] it was shown that if $A_\pm$ satisfy the relation

$$I = \int_{A_-}^{A_+} \left[(m - 2) \int_0^y g(s)\, ds\right]^{1/(m-2)} dy > 2, \tag{3.7}$$

then the gradient of the bounded solution blows-up in the interior of the domain in finite time for any initial compatible data. Note that according to [1] the relation $I \leqq 2$ does not guarantee the boundedness of the gradient. In order to simplify calculations we will consider the case when $g(u) = u$, $A_\pm = \pm A$, $u_0(x) = Ax$, where $A \geqq 0$. Hence (3.7) gives us

$$I = \int_{-A}^{A}\left[(m-2)\int_0^y s\,ds\right]^{1/(m-2)} dy$$

$$= \frac{((m-2)/2)^{\frac{1}{m-2}}}{1+2/(m-2)}[A^{1+2/(m-2)} - (-A)^{1+2/(m-2)}]$$

$$= 2\frac{((m-2)/2)^{\frac{1}{m-2}}}{1+2/(m-2)}A^{\frac{m}{m-2}}.$$

Thus (3.7) takes the form

$$A^{\frac{m}{m-2}} > \frac{1+2/(m-2)}{((m-2)/2)^{\frac{1}{m-2}}}.$$

It is clear that when $m \to \infty$, then $A \to 1$ from above. Thus we obtain that the sufficient condition for the existence of blow-up of the gradient for arbitrary $m$ is $A > 1$.

One can easily see that $f_2 = g(u)|u_x|^{m-1}u_x$ does not satisfy conditions (0.10), (0.11), so we can not guarantee that the blow-up of the gradient will not occur in the interior of the domain. For the function $v = u - Ax$ problem (3.5), (3.6) takes the form

$$v_t = v_{xx} + (v + Ax)|v_x + A|^{m-1}(v_x + A), \tag{3.8}$$

$$v(0,x) = v(t,\pm 1) = 0. \tag{3.9}$$

Consider now the representation $f_1 = (v + Ax)|v_x + A|^{m-1}(v_x + A)$, $f_2 = 0$. One can easily obtain the estimate $|u| \leqq A$. It is clear that

$$|(v + Ax)|v_x + A|^{m-1}(v_x + A)| \leqq A(|v_x| + A)^m.$$

Note that if in (0.14) we will take as $\psi(\rho) = A(|v_x| + A)^m$, then the global a priori estimate of $|v_x|$ and hence of $u_x$ will take place if

$$\int_0^{+\infty}\frac{\rho\,d\rho}{A(\rho + A)^m} = \int_0^{\infty}\frac{d\rho}{A(\rho + A)^{m-1}} - \int_0^{\infty}\frac{d\rho}{(\rho + A)^m}$$

$$= \frac{A^{1-m}}{m-2} - \frac{A^{1-m}}{m-1} = \frac{1}{(m-2)(m-1)A^{m-1}} \geqq 2A \geqq \operatorname{osc}(v).$$

So we have the global gradient estimate of a bounded solution of problem (3.5), (3.6) in the case $g(u) = u$, $u_0(x) = Ax$ when

$$A \leqq \frac{1}{[2(m-2)(m-1)]^{1/m}},$$

where $A \to 1$ from below when $m \to \infty$.

Consider the generalized KPZ (Kardar-Parisi-Zhang [17]) equation (see [4], [13], [18]) which arises in the evolution of the profile of growing interfaces

$$u_t = u_{xx} + \lambda |u_x|^\beta, \quad \beta \geqq 1,$$

where $\lambda$ is an arbitrary constant. From Lemmas 1, 2 it follows that the second and the third boundary value problems for the KPZ equation are solvable in the classical sense for any $\beta \geqq 0$.

Consider now the equation of curvature evolution for capillary surfaces

$$u_t = \frac{u_{xx}}{1 + u_x^2} + ku(1 + u_x^2)^{1/2}, \tag{3.10}$$

where $u(t, x)$ is a liquid surface and the constant $k$ is positive or negative according to whether the gravitation field is acting upward or downward. Obviously the Bernstein-Nagumo's condition is violated because

$$|ku(1 + p^2)^{1/2}| \leqq \frac{1}{1 + p^2} \psi(|p|) \quad \text{where } \psi(\rho) \equiv |k| \max|u|(1 + \rho^2)^{3/2}.$$

From Lemmas 1–3 it immediately follows that if $k \leqq 0$, then we have global gradient estimates for all boundary value problems. In fact, $f_2 = ku(1 + p^2)^{3/2}$ satisfies conditions (0.10)–(0.12). Moreover if $u_{0x}(x) \equiv 0$, then $u_x(t, x) \equiv 0$ (see (1.19), (1.21)).

In [2] it was shown that if $k$ is positive constant then for problem (3.10), (0.4), (0.5) the interior blow-up is possible to occur. Represent the right hand side as a sum of $f_1 = ku(1 + u_x^2)^{3/2}$ and $f_2 = 0$. Then from the results of Section 1 it follows that if

$$\int_0^{+\infty} \frac{\rho\, d\rho}{\psi(\rho)} = \int_0^{+\infty} \frac{\rho\, d\rho}{k \max|u|(1 + \rho^2)^{3/2}} > \operatorname{osc}(u),$$

i.e. $(kM)^{-1} > \operatorname{osc}(u)$, then we have a global estimate of the gradient. Here $M = \max|u|$. If $M = \operatorname{osc}(u)$, then this condition takes form $kM^2 < 1$. Note that $\max|u|$ stays bounded for any constant $k$.

Finally let us consider the Hamilton-Jacobi equation

$$u_t = f_2(t, x, u, u_x), \tag{3.11}$$

where $f_2$ satisfies conditions (0.10), (0.11) and (0.12). In order to obtain the viscosity solution of the Dirichlet problem for (3.11) (see [8]) consider the regularized problem:

$$u_t^\varepsilon = \varepsilon u_{xx}^\varepsilon + f_2(t, x, u^\varepsilon, u_x^\varepsilon). \tag{3.12}$$

For the solution of problem (3.12), (0.4), (0.5) the following estimates hold: $|u_x^\varepsilon| \leqq K = \max|u_0'(x)|$ (see (1.21)) and $|u^\varepsilon| \leqq \max|u_0|$. These estimates imply the Hölder continuity of $u^\varepsilon$ with respect to $t$:

$$|u^\varepsilon(t_1, x) - u^\varepsilon(t_2, x)| \leqq \tilde{C}|t_1 - t_2|^{1/2},$$

where the constant $\tilde{C}$ does not depend on $\varepsilon$ (see [19]). Hence there exists a subsequence $\varepsilon_n \to 0$ such that $u^{\varepsilon_n}$ converges uniformly to $u$. Passing to the limit when $\varepsilon_n \to 0$ in equation (3.12) we obtain a viscosity solution. Moreover $u_x^{\varepsilon_n} \to u_x$ *weakly in $L_\infty(Q_T)$ and, from equation (3.12), $u_t^{\varepsilon_n} \to u_t$ *weakly in $L_\infty(Q_T)$. Hence the obtained viscosity solution is Lipschitz continuous function and satisfies (3.11) almost everywhere. The other boundary value problems can be considered similarly.

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