Mathematical Notes, vol. 76, no. 4, 2004, pp. 546–557.
Translated from Matematicheskie Zametki, vol. 76, no. 4, 2004, pp. 592–603.

Dirichlet Problem for a Class of Quasilinear Elliptic Equations

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Received August 28, 2002

Abstract—In this paper, the Dirichlet problem for quasilinear elliptic equations is studied. New *a priori* estimates of the solution and its gradient are obtained. These estimates are derived without any assumptions on the smoothness of the coefficients and the right-hand side of the equation. Moreover, an arbitrary growth of the right-hand side with respect to the gradient of the solution is assumed. On the basis of the resulting estimates, existence theorems are proved.

Key words: *quasilinear elliptic equation, Dirichlet boundary-value problem, estimate of solutions, Kruzhkov's additional variable method.*

INTRODUCTION

The present paper is devoted to the study of the Dirichlet problem for quasilinear elliptic equations. As is well known (see [1, 2]), the proof of the solvability of boundary-value problems and, in particular, that of the Dirichlet problem, can be reduced to the derivation of an *a priori* estimate of the solution $u(\mathbf{x})$, $\mathbf{x}=(x_1,\dots,x_n)$, in the norm of the space $C^{1,\alpha}$, $\alpha\in(0,1)$ (we follow the notation used in [2]). This procedure is divided into four stages: estimating $\max|u(\mathbf{x})|$, estimating $\max|\nabla u(\mathbf{x})|$ on the boundary of the domain, carrying out a global estimate of $\max|\nabla u(\mathbf{x})|$ (i.e., on the whole domain), and, finally, estimating $|\nabla u(\mathbf{x})|$ in the norm of the space C^{α}, where $\nabla u=(u_{x_1},\dots,u_{x_n})$. The key role here is played by an estimate of $\max|\nabla u(\mathbf{x})|$, since after its derivation the solvability of the boundary-value problems follows without additional constraints on the character of nonlinearity of the equation. As is well known, the following classical approach for deriving the estimate of $\max|\nabla u(\mathbf{x})|$ is used in the general case (see [1–3]): the boundary estimate of $\max|\nabla u(\mathbf{x})|$ is obtained by constructing barriers for $u(\mathbf{x})$ near the boundary and the global estimate by differentiating and applying the maximum principle to the function $v(\mathbf{x})=|\nabla u(\mathbf{x})|^2$. This method dates back to Bernstein (see [4]). Obviously, such an approach requires the differentiability of the coefficients and the right-hand side of the equation as well as the imposition of a number of awkward constraints on the behavior of the first derivatives of the coefficients and of the right-hand side (see [1, 2]). To derive both a boundary and a global estimate of the gradient, it is necessary, in general, to require that Bernstein's condition [4] be satisfied (see [1–3]). This condition stipulates that the rate of growth of the right-hand side of the equation with respect to $|\nabla u(\mathbf{x})|$ as $|\nabla u(\mathbf{x})|\to+\infty$ must not must exceed the rate of growth of the principal part with respect to $|\nabla u(\mathbf{x})|$ by more than $|\nabla u(\mathbf{x})|^2$. It is well known (see, for example, [5]) that for the equation $\Delta u=f(\mathbf{x},u,\nabla u)$, where $f(\mathbf{x},u,\mathbf{p})$ is a continuous function, Bernstein's condition is sufficient to ensure that an estimate of $\max|u|$ yields an estimate of $\max|\nabla u|$. Pokhozhaev [6] showed that if the requirement of the continuity of the function f is replaced by the weaker condition $f\in L_q(\Omega)$, $q>n$, for $u\in W_q^2(\Omega)$, then Bernstein's condition is no longer sufficient for obtaining an estimate of $\max|\nabla u|$ from that of $\max|u|$. A condition on the growth of the function

 0001-4346/2004/7634-0546

$f(\mathbf{x}, u, \mathbf{p})$ with respect to $\mathbf{p}$, for which an estimate of $\max|u|$ yields an estimate of $\max|\nabla u|$, was stated in [6]. This condition depends on q and becomes Bernstein's condition for $q = +\infty$. As to an estimate of $\max|u(\mathbf{x})|$, then there exist a number of sufficient conditions guaranteeing such an estimate; see [1, 2, 7]. Note that parabolic equations were considered in [7]; however, the results obtained there can easily be carried over also to the elliptic case.

The present paper is devoted to the derivation of a boundary and a global *a priori* estimate of $\max|\nabla u(\mathbf{x})|$ for the case in which Bernstein's condition is not satisfied. To derive a global estimate, we use the additional variable method proposed by Kruzhkov (see [8]) for the study of quasilinear parabolic equations with one space variable (for details of this method, see also [9–14]). On the basis of Kruzhkov's method, one can obtain a global *a priori* estimate for $|\nabla u(\mathbf{x})|$ without differentiation for a particular class of quasilinear elliptic equations with many independent variables. The specifics of this class of equations consists, in particular, in the fact that the coefficients do not depend explicitly on the solution itself and on part of the space variables. We assume that the domain in which the Dirichlet problem is solved is convex. The main difference from a similar estimate in [12] is that we assume arbitrary growth of the right-hand side with respect to $|\nabla u(\mathbf{x})|$ independently of the growth of the principal part of the equation. As an example, we present the following problem:

$$\Delta u = f_1(\mathbf{x}) + \phi_1(u)\phi_2(\nabla u) \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{1}$$

Here $f_1(\mathbf{x})$ is a bounded function in $\overline{\Omega}$, $\phi_1(u)$ is a nondecreasing function, $u\phi_1(u) \ge 0$, and $\phi_2(\nabla u) \ge 0$. For example, $\phi_1 = u^3$ or $\phi_1 = u|u|^{1/3}$, while

$$\phi_2 = e^{|\nabla u|}\sqrt{|\nabla u|} + 1 \qquad \text{or} \qquad \phi_2 = |u_{x_1}|^{k_1} + \dots + |u_{x_n}|^{k_n} + k_0,$$

where the k_i are nonnegative real numbers.

As was shown by Ladyzhenskaya and Ural′tseva (see [1]), given estimates for $\max|u(\mathbf{x})|$ and $\max|\nabla u(\mathbf{x})|$, an estimate of $|\nabla u(\mathbf{x})|$ in the norm of the space C^α can be obtained under certain conditions on the smoothness of the coefficients and the right-hand side of the equation.

In Sec. 1, for simplicity, all the arguments are given for the equation $\Delta u = f(\mathbf{x}, u, \nabla u)$. Existence theorems based on the estimates obtained are given. In particular, the existence of a classical solution of problem (1) is guaranteed if $f_1 \in C^\alpha(\overline{\Omega})$, $\phi_1 \in C^\alpha(\mathbb{R})$, and $\phi_2 \in C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$.

Equations with two independent variables are studied in Sec. 2. There we consider a general two-dimensional equation and, moreover, we discuss the multidimensional quasilinear equation to which the results of Sec. 2 can be extended.

1. THE MULTIDIMENSIONAL CASE

For simplicity, all arguments are presented for $n = 3$; the case $n > 3$ can be studied in a similar way. Suppose that $\Omega \subset \mathbb{R}^3$ is a strictly convex domain and $\partial\Omega \in C^{2+\alpha}$, where $\alpha \in (0, 1)$. Without loss of generality, we assume that, first, the point $(0, 0, 0)$ belongs to Ω; second, the parts of $\partial\Omega$ lying in the half-spaces $x_1 \le 0$, $x_1 \ge 0$, can be expressed as

$$x_1 = F_1(x_2, x_3), \qquad x_1 = G_1(x_2, x_3),$$

while the parts of $\partial\Omega$ lying in the half-spaces $x_2 \le 0$, $x_2 \ge 0$ and $x_3 \le 0$, $x_3 \ge 0$, can be expressed as

$$x_2 = F_2(x_1, x_3), \quad x_2 = G_2(x_1, x_3) \qquad \text{and} \qquad x_3 = F_3(x_1, x_2), \quad x_3 = G_3(x_1, x_2),$$

respectively, where F_i and G_i are twice continuously differentiable functions; and, third, x_i changes between in the interval from $-l_i$ to l_i, $i = 1, 2, 3$.

Consider the following Dirichlet problem:

$$\Delta u = f(\mathbf{x}, u, \nabla u) \qquad \text{in} \quad \Omega \subset \mathbb{R}^3, \tag{1.1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.2}$$

where f can be expressed as

$$f(\mathbf{x}, u, \mathbf{p}) = f_1(\mathbf{x}, u, \mathbf{p}) + f_2(\mathbf{x}, u, \mathbf{p}). \tag{1.3}$$

Suppose that for $\mathbf{x} \in \Omega$, $|u| \le M$ and finite $\mathbf{p} = (p_1, p_2, p_3)$ the function f_1 satisfies the structural constraint

$$|f_1(\mathbf{x}, u, \mathbf{p})| < \psi(|p_1|), \tag{1.4}$$

where $\psi(\rho) > 0$ is a continuously differentiable function such that

$$\int_0^{+\infty} \frac{\rho\, d\rho}{\psi(\rho)} > \mu_1 \equiv \max\{M, \operatorname{osc}(u)\}, \tag{1.5}$$

while the function f_2 satisfies the relation

$$u f_2(\mathbf{x}, u, \mathbf{p}) \ge 0. \tag{1.6}$$

We introduce the functions $h_1(x_1)$ and $h_2(x_1) \equiv h_1(-x_1)$:

$$h_1'' + \psi(|h_1'|) = 0, \qquad h_1(-l_1) = 0, \qquad h_1(-l_1 + \tau_0) = \mu_1;$$

the constant τ_0 will be chosen below. Let us express h_1 in parametric form:

$$h_1 = h_1(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \qquad x_1 = x_1(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)} - l_1,$$

where the parameter q lies in the interval $[q_0, q_1]$, with $q_1 > q_0 > 0$ and

$$h_1(q_0) = \int_{q_0}^{q_1} \frac{\rho\, d\rho}{\psi(\rho)} = \mu_1.$$

This is possible by condition (1.5). Let

$$\tau_0 = \int_{q_0}^{q_1} \frac{d\rho}{\psi(\rho)}. \tag{1.7}$$

Obviously, $h_1' > 0$ and $h_2' < 0$.

Lemma 1.1. *Suppose that $u(\mathbf{x})$ is a classical solution of problem* (1.1), (1.2) *such that $|\nabla u| < +\infty$ in $\overline{\Omega}$. Suppose that conditions* (1.3), (1.4), *and* (1.6) *are valid. Then*

$$|u(\mathbf{x})| \le h_k(\zeta_k) \quad \forall \mathbf{x} \in \overline{D}_k, \qquad k = 1, 2,$$

where $\zeta_1 = x_1 - F_1 - l_1$, $\zeta_2 = x_1 - G_1 + l_1$, and

$$D_1 = \{\mathbf{x} \colon F_1(x_2, x_3) < x_1 < F_1(x_2, x_3) + \tau_0,\ (x_2, x_3) \in \Omega_1\} \cap \Omega,$$
$$D_2 = \{\mathbf{x} \colon G_1(x_2, x_3) - \tau_0 < x_1 < G_1(x_2, x_3),\ (x_2, x_3) \in \Omega_1\} \cap \Omega;$$

here Ω_1 is the projection of the domain Ω on the plane (x_2, x_3).

Proof. If $\tau_0 \geq 2l_1$, then $\partial D_1 = \partial D_2 = \partial\Omega$ and, by the boundary condition, $u|_{\partial\Omega} = 0 \leq h_k$, $k = 1, 2$.

If $\tau_0 < 2l_1$, then ∂D_1 consists of two parts: Γ_1 and Γ_2, where $\Gamma_1 \subset \partial\Omega$, and Γ_2 is the part of the surface $x_1 = F_1(x_2, x_3) + \tau_0$ that lies in Ω. On Γ_1, we have $u = 0 \leq h_1$, while on Γ_2 we obtain

$$h_1(-l_1 + \tau_0) = \mu_1 \geq u.$$

Similarly, for $\tau_0 \leq 2l_1$ the surface $\partial D_2 = \Gamma_3 \cup \Gamma_4$, where $\Gamma_3 \subset \partial\Omega$, while Γ_4 is the part of the surface $x_1 = G_1(x_2, x_3) - \tau_0$ that lies in Ω. On Γ_3, we have $u = 0 \leq h_2$, while on Γ_4 we obtain $h_2(l_1 - \tau_0) = \mu_1 \geq u$. Therefore, $|u(\mathbf{x})| \leq h_k(\zeta_k)$ on ∂D_k.

In D_k, we have

$$\Delta h_k(\zeta_k) = h_k'' + h_{kx_2x_2} + h_{kx_3x_3} \leq -\psi(|h_k'|), \qquad k = 1, 2,$$

since for $i = 2, 3$ the following relations hold:

$$h_{1x_ix_i}(\zeta_1) = -(F_{1x_i}^2 \psi(|h_1'(x_1 - F_1 - l_1)|) + F_{1x_ix_i} h_1'(x_1 - F_1 - l_1)) \leq 0,$$
$$h_{2x_ix_i}(\zeta_2) = -(G_{1x_i}^2 \psi(|h_2'(x_1 - G_1 + l_1)|) + G_{1x_ix_i} h_2'(x_1 - G_1 + l_1)) \leq 0.$$

This can readily be verified if we recall that $h_1' \geq 0$ and $h_2' \leq 0$, and, moreover, by the convexity of the domain, we have $F_{1x_ix_i} \geq 0$, $G_{1x_ix_i} \leq 0$. Therefore, for $v \equiv u - h_k$ we obtain

$$\Delta v \geq f(\mathbf{x}, u, \nabla u) + \psi(|h_k'|). \tag{1.8}$$

If the function v attains its maximum at the point $N \in \overline{D}_k \setminus \partial D_k$, then, at this point, $v > 0$, $v_{x_i} = 0$, i.e., $u > h_k \geq 0$, $\nabla u = \nabla h_k$; therefore, Eqs. (1.4), (1.6), and (1.8) imply

$$\Delta v|_N \geq f_1(\mathbf{x}, u, \nabla u) + \psi(|u_{x_1}|) + f_2(\mathbf{x}, u, \nabla u)|_N > 0.$$

This contradicts the assumption that the function v attains its maximum at the interior points of the domain D_k. Therefore,

$$u(\mathbf{x}) \leq h_k(\zeta_k) \quad \text{in} \quad \overline{D}_k, \qquad k = 1, 2.$$

Next, we obtain a lower bound. For $w \equiv u + h_k$, we have

$$\Delta w \leq f(\mathbf{x}, u, \nabla u) - \psi(|h_k')|). \tag{1.9}$$

If the function w reaches its minimum at the point $N_1 \in \overline{D}_k \setminus \partial D_k$, then, at this point, $u < 0$ and $\nabla u = -\nabla h_k$, and, therefore, by (1.4) and (1.6), from (1.9) we obtain

$$\Delta w|_{N_1} \leq f_1(\mathbf{x}, u, \nabla u) - \psi(|u_{x_1}|) + f_2(\mathbf{x}, u, \nabla u)|_{N_1} < 0.$$

This contradicts the assumption that w attains its minimum at the interior points of the domain D_k. Therefore, $u(\mathbf{x}) \geq -h_k(\zeta_k)$ in $\overline{D}_k$, $k = 1, 2$. The lemma is proved. $\square$

Now, we derive a global estimate of $|u_{x_1}|$. In addition, suppose that f_2 satisfies the relations

$$f_2(x_1, x_2, x_3, u, p_1, p_2, p_3) - f_2(\zeta, x_2, x_3, v, p_1, p_2, p_3) \geq 0, \tag{1.10_1}$$
$$f_2(\zeta, x_2, x_3, u, -p_1, p_2, p_3) - f_2(x_1, x_2, x_3, v, -p_1, p_2, p_3) \geq 0 \tag{1.10_2}$$

for $x_1 > \zeta$, $u > v$, $p_1 > 0$, and any p_2, p_3.

Lemma 1.2. *Suppose that the assumptions of Lemma* 1.1 *and conditions* (1.10) *hold. Then* $u(\mathbf{x})$ *satisfies the estimate*

$$\sup_{\overline{\Omega}} |u_{x_1}| \le C_1,$$

where the constant C_1 *depends only on* ψ *and* M.

Proof. Consider Eq. (1.1) at two different points of the domain Ω, namely, $\mathbf{x} = (x_1, x_2, x_3)$ and (ζ, x_2, x_3). We have

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} = f^{(x_1)} \qquad \text{for} \quad u = u(\mathbf{x}), \tag{1.11}$$

$$u_{\zeta\zeta} + u_{x_2x_2} + u_{x_3x_3} = f^{(\zeta)} \qquad \text{for} \quad u = u(\zeta, x_2, x_3), \tag{1.12}$$

where $f^{(z)} \equiv f(z, x_2, x_3, u(z, x_2, x_3), \nabla u(z, x_2, x_3))$. Subtracting Eq. (1.12) from Eq. (1.11), for the function $v(\mathbf{x}, \zeta) \equiv u(x_1, x_2, x_3) - u(\zeta, x_2, x_3)$ we obtain

$$\Delta_{\mathbf{x},\zeta} v \equiv v_{x_1x_1} + v_{\zeta\zeta} + v_{x_2x_2} + v_{x_3x_3} = f^{(x_1)} - f^{(\zeta)}.$$

Suppose that the function $h(\tau) \equiv h(x_1 - \zeta)$ is a solution of the problem

$$h'' + \psi(|h'|) = 0, \qquad h(0) = 0, \qquad h(\tau_0) = \mu_1;$$

the quantity τ_0 was defined in the construction of the barrier h_1 (see (1.7)). Obviously,

$$\Delta_{\mathbf{x},\zeta} h(x_1 - \zeta) = 2h'' = -2\psi(|h'|).$$

For the function $w \equiv v - h(x_1 - \zeta)$, we obtain

$$\begin{aligned}
\Delta_{\mathbf{x},\zeta} w &= f^{(x_1)} - f^{(\zeta)} - 2\psi(|h'|) \\
&= f_1(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) - \psi(|h'|) \\
&\quad - (f_1(\zeta, x_2, x_3, u(\zeta, x_2, x_3), \nabla u(\zeta, x_2, x_3)) - \psi(|h'|)) \\
&\quad + f_2(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x})) - f_2(\zeta, x_2, x_3, u(\zeta, x_2, x_3), \nabla u(\zeta, x_2, x_3)).
\end{aligned} \tag{1.13}$$

Consider the domain Q:

$$Q = \{(\mathbf{x}, \zeta) \colon x_1 \in (F_1, G_1),\ \zeta \in (F_1, G_1),\ 0 < x_1 - \zeta < \tau_0,\ (x_2, x_3) \in \Omega_1\}$$

(recall that Ω_1 is the projection of Ω on the plane (x_2, x_3)). If $\tau_0 \ge 2l_1$, then we assume

$$Q = \{(\mathbf{x}, \zeta) \colon x_1 \in (F_1, G_1),\ \zeta \in (F_1, G_1),\ 0 < x_1 - \zeta,\ (x_2, x_3) \in \Omega_1\}.$$

Suppose that the function w takes the largest value at an interior point N of the domain Q ($N \in \overline{Q} \setminus \partial Q$). Obviously, at the point N, we have $w > 0$, $\nabla w = 0$ and, therefore, $u \ge 0$, $u_{x_1}(\mathbf{x}) = h'(x_1 - \zeta)$, $u_\zeta(\zeta, x_2, x_3) = h'(x_1 - \zeta)$, $u_{x_i}(\mathbf{x}) = u_{x_i}(\zeta, x_2, x_3)$, $i = 2, 3$, i.e.,

$$\nabla u(\mathbf{x})|_N = \nabla u(\zeta, x_2, x_3)|_N.$$

Thus, from (1.4), (1.10_1) and (1.13) we obtain $\Delta_{\mathbf{x},\zeta} w|_N > 0$, which contradicts the assumption that w attains its maximum at an interior point of the domain Q.

Let us show that $w \le 0$ on ∂Q.

(1) For $x_1 = \zeta$, we have $v = h = 0$.

(2) For $x_1 - \zeta = \tau_0$, we obtain $h(\tau_0) = \mu_1 \geq u(\mathbf{x}) - u(\zeta, x_2, x_3)$. This part of the boundary is present only in the case $\tau_0 < 2l_1$.

(3) For $\zeta = F_1(x_2, x_3)$, $x_1 \in [F_1, F_1 + \tau_0] \cap [F_1, G_1]$, $(x_2, x_3) \in \Omega_1$ the relation

$$v - h = u(\mathbf{x}) - h(x_1 - F_1)$$

holds. Let us now show that $u(\mathbf{x}) \leq h(x_1 - F_1)$. To do this, it suffices to obtain the equality $h(x_1 - F_1) = h_1(x_1 - F_1 - l_1)$, and then apply Lemma 1.1. The required equality immediately follows from the relations

$$h_1''(\zeta_1) + \psi(|h_1'(\zeta_1)|) = 0, \qquad h_1(-l_1) = 0, \qquad h_1(-l_1 + \tau_0) = \mu_1, \qquad \zeta_1 = x_1 - F_1 - l_1,$$
$$h''(\eta) + \psi(|h'(\eta)|) = 0, \qquad h(0) = 0, \qquad h(\tau_0) = \mu_1, \qquad \eta = x_1 - F_1.$$

(4) For $x_1 = G_1(x_2, x_3)$, $\zeta \in [G_1 - \tau_0, G_1] \cap [F_1, G_1]$, $(x_2, x_3) \in \Omega_0$, we obtain

$$v - h = -u(\zeta, x_2, x_3) - h(G_1 - \zeta).$$

Let us show that $u(\zeta, y) \geq -h(G_1(y) - \zeta)$. It suffices to verify $h(G_1 - \zeta) = h_2(\zeta - G_1 + l_1)$, and then again apply Lemma 1.1. The last equality immediately follows that from the relations

$$h_2''(\zeta_2) + \psi(|h_2'(\zeta_2)|) = 0, \qquad h_2(l_1 - \tau_0) = \mu_1, \qquad h_2(l_1) = 0, \qquad \zeta_2 = \zeta - G_1 + l_1,$$
$$h''(\tilde\eta) + \psi(|h'(\tilde\eta)|) = 0, \qquad h(0) = 0, \qquad h(\tau_0) = \mu_1, \qquad \tilde\eta = G_1 - \zeta.$$

Thus, $w \leq 0$ on ∂Q. In view of the fact that w cannot to attain its maximum in $\overline{Q} \setminus \partial Q$ we find that

$$u(\mathbf{x}) - u(\zeta, x_2, x_3) \leq h(x_1 - \zeta) \qquad \text{in} \quad \overline{Q}.$$

Similarly, taking the function $\tilde v \equiv u(\zeta, x_2, x_3) - u(\mathbf{x})$ instead of v, we obtain $v \geq -h(x_1 - \zeta)$ in $\overline{Q}$ (here we have used condition (1.10_2)).

By the symmetry of the variables x_1 and ζ, we consider the case $\zeta > x_1$ in the same way. As a result, we see that for $x_1 \in [F_1, G_1]$, $\zeta \in [F_1, G_1]$, $(x_2, x_3) \in \Omega_1$, $0 < |x_1 - \zeta| < \tau_0$ the following inequality holds:

$$\frac{|u(\mathbf{x}) - u(\zeta, x_2, x_3)|}{|x_1 - \zeta|} \leq \frac{h(|x_1 - \zeta|) - h(0)}{|x_1 - \zeta|},$$

which, in turn, implies the estimate $|u_{x_1}(\mathbf{x})| \leq h'(0) = C_1$. The lemma is proved. $\square$

Next, let us state conditions ensuring the existence of an *a priori* estimate of u_{x_2}.

Suppose that, for $\mathbf{x} \in \Omega$, $|u| \leq M$, $|p_1| \leq C_1$, and finite p_2, p_3, the function f_1 satisfies the inequality

$$|f_1(\mathbf{x}, u, \mathbf{p})| < \psi(|p_2|). \tag{1.14}$$

Let us introduce the functions $h_3(x_2)$, $h_4(x_2)$ as follows:

$$h_4(x_2) = h_3(-x_2), \qquad h_3'' + \psi(|h_3'|) = 0, \qquad h_3(-l_2) = 0, \quad h_3(-l_2 + \tau_0) = \mu_1.$$

The proof of the following lemma is similar to that of Lemma 1.1.

Lemma 1.3. *Suppose that the assumptions of Lemmas* 1.1, 1.2 *and condition* (1.14) *are valid. Then*

$$|u(\mathbf{x})| \leq h_k(\zeta_k) \quad \forall \mathbf{x} \in \overline{D}_k, \qquad k = 3, 4,$$

where $\zeta_3 = x_2 - F_2 - l_2$, $\zeta_4 = x_2 - G_2 + l_2$,

$$D_3 = \{\mathbf{x}\colon F_2 < x_2 < F_2 + \tau_0, (x_1, x_3) \in \Omega_2\} \cap \Omega,$$
$$D_4 = \{\mathbf{x}\colon G_2 - \tau_0 < x_2 < G_2, (x_1, x_3) \in \Omega_2\} \cap \Omega,$$

and Ω_2 *is the projection of* Ω *on the plane* (x_1, x_3).

In addition, suppose that

$$f_2(x_1, x_2, x_3, u, p_1, p_2, p_3) - f_2(x_1, \zeta, x_3, v, p_1, p_2, p_3) \ge 0, \tag{1.15_1}$$
$$f_2(x_1, \zeta, x_3, u, p_1, -p_2, p_3) - f_2(x_1, x_2, x_3, v, p_1, -p_2, p_3) \ge 0, \tag{1.15_2}$$

for $x_2 > \zeta$, $u > v$, $p_2 > 0$, and any p_1, p_3.

Lemma 1.4. *Suppose that conditions* (1.15) *and the assumptions of Lemmas* 1.1–1.3 *are valid. Then* $u(\mathbf{x})$ *satisfies the estimate*

$$\sup_{\overline{\Omega}} |u_{x_2}| \le C_2,$$

where the constant C_2 *depends only on* ψ *and* M.

The proof of this lemma is similar to that of Lemma 1.2.

Let us pass to an estimate of u_{x_3}. Suppose that for $\mathbf{x} \in \Omega$, $|u| \le M$, $|p_1| \le C_1$, $|p_2| \le C_2$, and finite p_3, the function f_1 satisfies the condition

$$|f_1(\mathbf{x}, u, \mathbf{p})| < \psi(|p_3|). \tag{1.16}$$

Let us introduce the functions $h_5(x_3)$, $h_6(x_3)$ as follows:

$$h_6(x_3) = h_5(-x_3), \qquad h_5'' + \psi(|h_5'|) = 0, \qquad h_5(-l_3) = 0, \quad h_5(-l_3 + \tau_0) = \mu_1.$$

The proof of the following lemma is similar to that of Lemma 1.1.

Lemma 1.5. *Suppose that the assumptions of Lemmas* 1.1–1.4 *and condition* (1.16) *are valid. Then*

$$|u(\mathbf{x})| \le h_k(\zeta_k) \quad \forall \mathbf{x} \in \overline{D}_k, \qquad k = 5, 6,$$

where $\zeta_5 = x_3 - F_3 - l_3$, $\zeta_6 = x_3 - G_3 + l_3$,

$$D_5 = \{\mathbf{x}\colon F_3 < x_3 < F_3 + \tau_0,\ (x_1, x_2) \in \Omega_3\} \cap \Omega,$$
$$D_6 = \{\mathbf{x}\colon G_3 - \tau_0 < x_3 < G_3,\ (x_1, x_2) \in \Omega_3\} \cap \Omega,$$

and Ω_3 *is the projection of* Ω *on the plane* (x_1, x_2).

To obtain a global estimate of u_{x_3}, we assume that

$$f_2(x_1, x_2, x_3, u, p_1, p_2, p_3) - f_2(x_1, x_2, \zeta, v, p_1, p_2, p_3) \ge 0, \tag{1.17_1}$$
$$f_2(x_1, x_2, \zeta, u, p_1, p_2, -p_3) - f_2(x_1, x_2, x_3, v, p_1, p_2, -p_3) \ge 0 \tag{1.17_2}$$

for $x_3 > \zeta$, $u > v$, $p_3 > 0$, and any p_1, p_2.

Lemma 1.6. *Suppose that conditions* (1.17) *and the assumptions of Lemmas* 1.1–1.5 *are valid. Then* $u(\mathbf{x})$ *satisfies the estimate*

$$\sup_{\overline{\Omega}} |u_{x_3}| \le C_3,$$

where the constant C_3 *depends only on* ψ *and* M.

The proof of this lemma is similar to that of Lemma 1.2.

Remark 1.1. In Lemmas 1.2, 1.4, and 1.6, we have obtained estimates of $|u_{x_i}|$ depending only on $M = \max|u|$ and ψ, provided that ψ satisfies condition (1.5). It is readily verified that if condition (1.5) is replaced by the following one:

$$\int_0^{+\infty} \frac{d\rho}{\psi(\rho)} > 2l_i, \tag{1.18}$$

then we can find estimates of $|u_{x_i}|$ depending only on ψ and l_i. To do this, it is necessary to introduce some changes in the construction of the barriers $h_i(x_i)$ and $h(\tau)$. Namely, h_i and h must be the solutions of the problems

$$\begin{gathered} h_i'' + \psi(|h_i'|) = 0, \quad h_i(-l_i) = 0, \quad h_i(l_i) = H, \qquad i = 1, 3, 5, \\ h'' + \psi(|h'|) = 0, \qquad h(0) = 0, \qquad h(2l_i) = H, \end{gathered} \tag{1.19}$$

respectively. Let us express the solution of problem (1.19) in parametric form:

$$h(q) = \int_q^{q_1} \frac{\rho\, d\rho}{\psi(\rho)}, \qquad \tau(q) = \int_q^{q_1} \frac{d\rho}{\psi(\rho)}.$$

The parameter $q \in [q_0, q_1]$, and q_0, q_1 are chosen so that $0 \le q_0 < q_1 < +\infty$ and $\tau(q_0) = 2l_1$, which is possible in view of (1.18). The estimate

$$|u(\mathbf{x})| \le h_k(\zeta_k), \qquad k = 1, \dots, 6,$$

is obtained just as in Lemma 1.1; note that in this case $\tau_0 \equiv \tau(q_0) = 2l_i$ and $D_i = \Omega$, $i = 1, \dots, 6$. A global estimate is established just as in Lemma 1.2.

Let us state an existence and uniqueness theorem.

Theorem 1.1. *Suppose that $f \in C^\alpha(\Omega \times \mathbb{R} \times \mathbb{R}^3)$ for some $\alpha \in (0,1)$ and conditions* (1.3), (1.4), (1.6), (1.10), *and* (1.14)–(1.17) *are satisfied. Also, suppose that a condition ensuring the a priori estimate $|u| \le M$ holds. Then, in any strictly convex domain Ω such that $\partial\Omega \in C^{2,\alpha}$, problem* (1.1), (1.2) *has at least one solution $u(\mathbf{x}) \in C^{2,\alpha}(\overline{\Omega})$. If the function f is increasing in the variable u, then such a solution is unique.*

Proof. The existence of such a solution follows from the *a priori* estimates obtained above and Theorem 13.8 from [2]. In [2, Theorem 10.2], the uniqueness of the classical solution is proved under the assumption that the function a_{ij} is independent from u, a_{ij} and f are differentiable with respect to $\mathbf{p}$, and the function f is decreasing in the variable u. Proceeding in the same way as in the proof of Theorem 2.1 from [13], we can easily dispense with the differentiability condition with respect to $\mathbf{p}$.

Let us present two examples. First, we consider problem (1) stated in the Introduction. An estimate of $\max|u|$ can be obtained, for example, from [7]. It is readily seen that conditions (1.4), (1.6), and (1.10), (1.14)–(1.17) are satisfied. Thus, if

$$f_1(\mathbf{x}) \in C^\alpha(\overline{\Omega}), \qquad f_2 \equiv \phi_1(u)\phi_2(\nabla u) \in C^\alpha(\mathbb{R}^4),$$

then Theorem 1.1 guarantees the existence of a solution of problem (1) belonging to the space $C^{2,\alpha}(\overline{\Omega})$.

Next, consider the following equation:

$$\Delta u = f_1(\mathbf{x}, u, u_{x_1}) + f_2(\mathbf{x}, u, \nabla u), \tag{1.20}$$

where $|f_1(\mathbf{x}, u, p_1)| \le \widetilde{K}(1+p_1^2)$ for $\mathbf{x} \in \Omega$, $|u| \le M$ and any p_1. For f_2 we can take the same function as in the previous example. For a classical solution of problem (1.20), (1.2) such that $|\nabla u| < +\infty$, the estimates given in Lemmas 1.1–1.6 are valid. Condition (1.4) holds with $\psi(p_1) = K_1(1+p_1^2)$, where $K_1 > \widetilde{K}$. The estimate $|u_{x_1}| \le C_1$ is a consequence of Lemma 1.2 Conditions (1.14), (1.16) are satisfied with

$$\psi \equiv K_2 > \max\{\widetilde{K}, F\}, \qquad \text{where} \quad F = \max |f_1(\mathbf{x}, u, p_1)|,$$

the maximum of the function $|f_1(\mathbf{x}, u, p_1)|$, is taken over the set $\overline{\Omega} \times [-M, M] \times [-C_1, C_1]$. Lemmas 1.4 and 1.6 yield the estimates $|u_{x_2}| \le C_2$, $|u_{x_3}| \le C_3$.

If in Eq. (1.20) $f_1 \in C^\alpha(\Omega \times \mathbb{R} \times \mathbb{R})$ and $f_2 \in C^\alpha(\Omega \times \mathbb{R} \times \mathbb{R}^3)$, then, by Theorem 1.1, there exists a solution of problem (1.20), (1.2) belonging to $C^{2,\alpha}(\overline{\Omega})$. $\square$

2. THE TWO-DIMENSIONAL CASE

Consider the following Dirichlet problem:

$$a(x, y, u, \nabla u)u_{xx} + 2b(x, y, u, \nabla u)u_{xy} + c(x, y, u, \nabla u)u_{yy} = f(x, y, u, \nabla u) \qquad \text{in} \quad \Omega \subset \mathbb{R}^2, \tag{2.1}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{2.2}$$

where $\nabla u = (u_x, u_y)$. The functions a, b, c, and f are defined on the set $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2$ and take finite values for $(x, y) \in \Omega$ and finite u, ∇u. Suppose that

$$a \ge a_0 > 0, \quad c \ge c_0 > 0, \quad b^2 - ac < 0 \qquad \text{in} \quad \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2, \tag{2.3}$$

where a_0 and c_0 are constants. We assume that f can be expressed as (1.3), where

$$|f_1(x, y, u, \mathbf{p})| < a(x, y, u, \mathbf{p})\psi(|p_1|) \tag{2.4}$$

for $(x, y) \in \Omega$, $|u| \le M$, and finite $\mathbf{p} = (p_1, p_2)$. The function ψ was defined in Sec. 1 (see (1.5)).

Lemma 2.1. *Suppose that $u(x, y)$ is a classical solution of problem* (2.1), (2.2) *for which $|\nabla u| < +\infty$ in $\overline{\Omega}$ and conditions* (2.3), (2.4), *and* (1.6) *are valid. Suppose that*

$$b(x, y, u, \nabla h_k(\zeta_k))h_{kx_ix_j}(\zeta_k) \le 0, \quad k = 1, 2, \qquad b(x, y, u, \mathbf{p}) = b(x, y, u, -\mathbf{p}). \tag{2.5}$$

Then

$$|u(x, y)| \le h_k(\zeta_k) \quad \forall (x, y) \in \overline{D}_k, \qquad k = 1, 2,$$

where

$$D_1 = \{(x, y)\colon F_1(y) < x < F_1(y) + \tau_0,\ -l_2 < y < l_2)\} \cap \Omega,$$
$$D_2 = \{(x, y)\colon G_2(y) - \tau_0 < x < G_1(y),\ -l_2 < y < l_2\} \cap \Omega.$$

The proof of this lemma is similar to that of Lemma 1.1. The functions h_k, ζ_k were defined in Sec. 2.

We now obtain a global estimate $|u_x|$. Suppose that

$$f_3(x, y, u, p_1, p_2) - f_3(\zeta, y, v, p_1, p_2) \ge 0, \tag{2.6$_1$}$$

$$f_3(\zeta, y, u, -p_1, p_2) - f_3(x, y, v, -p_1, p_2) \ge 0 \tag{2.6$_2$}$$

for $x \ge \zeta$, $u \ge v$, $p_1 \ge 0$, and any p_2, $f_3 = f_2/c$.

Lemma 2.2. *Suppose that the assumptions of Lemma* 2.1, *condition* (2.2), *and the inequality*

$$2b^2 - ac < 0 \qquad in \quad \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^1 \tag{2.7}$$

are valid. Then $u(x,y)$ *satisfies the estimate*

$$\sup_{\overline{\Omega}} |u_x| \le C_4,$$

where the constant C_4 *depends only on* ψ *and* M.

Proof. Let us rewrite (2.1) as follows:

$$\frac{a^{(x)}}{c^{(x)}} u_{xx} + 2\frac{b^{(x)}}{c^{(x)}} u_{xy} + u_{yy} = \frac{f^{(x)}}{c^{(x)}}, \qquad \text{where} \quad g^{(z)} \equiv g(z, y, u(z,y), \nabla u(z,y)).$$

Consider this equation at the point $(x, \zeta) \in \Omega$, $\zeta \ne x$:

$$\frac{a^{(\zeta)}}{c^{(\zeta)}} u_{\zeta\zeta} + 2\frac{b^{(\zeta)}}{c^{(\zeta)}} u_{\zeta y} + u_{yy} = \frac{f^{(\zeta)}}{c^{(\zeta)}}.$$

Subtracting this equation from the equation for $u(x,y)$, for the function

$$v(x, y, \zeta) \equiv u(x,y) - u(\zeta, y),$$

we obtain

$$L^* v \equiv \frac{a^{(x)}}{c^{(x)}} v_{xx} + 2\frac{b^{(x)}}{c^{(x)}} v_{xy} + \frac{a^{(\zeta)}}{c^{(\zeta)}} v_{\zeta\zeta} + 2\frac{b^{(\zeta)}}{c^{(\zeta)}} v_{\zeta y} + v_{yy} = \frac{f^{(x)}}{c^{(x)}} - \frac{f^{(\zeta)}}{c^{(\zeta)}}.$$

Condition (2.7) ensures the ellipticity of the operator L^*. Indeed,

$$\det \begin{pmatrix} \dfrac{a^{(x)}}{c^{(x)}} & \dfrac{b^{(x)}}{c^{(x)}} & 0 \\ \dfrac{b^{(x)}}{c^{(x)}} & 1 & \dfrac{b^{(\zeta)}}{c^{(\zeta)}} \\ 0 & \dfrac{b^{(\zeta)}}{c^{(\zeta)}} & \dfrac{a^{(\zeta)}}{c^{(\zeta)}} \end{pmatrix} = \frac{1}{2}\left(\frac{a^{(x)}}{c^{(x)}} \left[\frac{a^{(\zeta)}}{c^{(\zeta)}} - 2\left(\frac{b^{(\zeta)}}{c^{(\zeta)}}\right)^2 \right] + \frac{a^{(\zeta)}}{c^{(\zeta)}} \left[\frac{a^{(x)}}{c^{(x)}} - 2\left(\frac{b^{(x)}}{c^{(x)}}\right)^2 \right] \right) > 0.$$

The end of the proof is similar to that of Lemma 1.2. $\square$

Suppose that for $(x,y) \in \Omega$, $|u| \le M$, $|p_1| \le C_4$, and finite p_2 the following inequality holds:

$$|f_1(x, y, u, p_1, p_2)| < a(x, y, u, p_1, p_2)\psi(|p_2|). \tag{2.8}$$

Lemma 2.3. *Suppose that* $u(x,y)$ *is a classical solution of problem* (2.1), (2.2) *for which* $|\nabla u| < +\infty$ *in* $\overline{\Omega}$. *Suppose that conditions* (2.3)–(2.8) *are valid. Then*

$$|u(x,y)| \le h_k(\zeta_k) \quad \forall (x,y) \in \overline{D}_k, \qquad k = 3, 4,$$

where

$$D_3 = \{(x,y)\colon F_2(x) < y < F_2(x) + \tau_0,\ -l_1 < x < l_1\} \cap \Omega,$$
$$D_4 = \{(x,y)\colon G_2 - \tau_0 < y < G_2,\ -l_1 < x < l_1)\} \cap \Omega.$$

The proof of this lemma is similar to that of Lemma 1.1.

Let us state the conditions guaranteeing the existence of a global estimate of $|u_y|$. Suppose that

$$f_4(x, y, u, p_1, p_2) - f_4(x, \zeta, v, p_1, p_2) \ge 0, \tag{2.9_1}$$
$$f_4(x, \zeta, u, p_1, -p_2) - f_4(x, y, v, p_1, -p_2) \ge 0 \tag{2.9_2}$$

for $y \ge \zeta$, $u \ge v$, $p_2 \ge 0$, and any p_1, $f_4 = f_2/a$.

Lemma 2.4. *Suppose that the assumptions of Lemma 2.3 and condition (2.9) are valid. Then $u(x,y)$ satisfies the estimate*

$$\sup_{\overline{\Omega}} |u_y| \le C_5\,,$$

where the constant C_5 depends only on ψ and M.

The proof of Lemma 2.4 is similar to that of Lemma 2.1.

Let us state the following existence theorem.

Theorem 2.1. *Suppose that $a, b, c, f \in C^{\alpha}(\Omega\times\mathbb{R}\times\mathbb{R}^2)$ for some $\alpha \in (0,1)$ and conditions (2.3)–(2.9) hold. Also, suppose that a condition guaranteeing the estimate $|u| \le M$ is satisfied. Then, in any strictly convex domain Ω, problem (2.1), (2.2) has at least one solution $u(x,y) \in C^{2,\alpha}(\overline{\Omega})$. If the function f is increasing in the variable u, and the functions a, b, c are independent of u, then such a solution is unique.*

Proof. In view of the estimates obtained above and conditions (2.3), Eq. (2.1) can be regarded as a linear strictly elliptic equation

$$\tilde{a}(x,y)u_{xx} + 2\tilde{b}(x,y)u_{xy} + \tilde{c}(x,y)u_{yy} = \tilde{f}(x,y)$$

with bounded coefficients $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, and bounded right-hand side of $\tilde{f}$. It follows from Theorem 11.4 [2] that the solution of the Dirichlet problem for this equation (with smooth boundary conditions and smooth boundary) satisfies the inequality

$$|u|_{C^{1+\delta}(\overline{\Omega})} \le C\left(\sup_{\Omega}|u| + \sup_{\Omega}\left(\frac{|\tilde{f}|}{\lambda}\right)\right), \qquad \delta \in (0,1),$$

where λ is the minimal eigenvalue of the matrix of leading coefficients. The constant C depends only on the upper bounds for the moduli of the coefficients and the right-hand side. It follows from this estimate and Theorem 13.8 [2] that the required solution exists. $\square$

In conclusion, we present an equation to which the results from Sec. 1. can be extended. Consider the Dirichlet problem

$$a_{ij}(x_i, x_j, \nabla u)u_{x_i x_j} = f(\mathbf{x}, u, \nabla u) \quad \text{in} \quad \Omega \subset \mathbb{R}^3, \qquad u = 0 \quad \text{on} \quad \partial\Omega\,; \tag{2.10}$$

here $a_{ii} = a_{ii}(x_i, \nabla u)$. We assume that for $\mathbf{x} \in D_k$, $k = 1, 2$, $|u| \le M$, and arbitrary $\mathbf{p}$, the following inequality holds:

$$\sum_{i,j=1,\,i\neq j}^{3} a_{ij}(x_i, x_j, \nabla h_k(\zeta_k))h_{k x_i x_j}(\zeta_k) \le 0,$$
$$a_{ij}(x_i, x_j, \mathbf{p}) = a_{ij}(x_i, x_j, -\mathbf{p}) \qquad \text{for} \quad i \neq j,$$

this is the multidimensional analog of condition (2.5). Moreover, it is necessary to require the validity of the multidimensional analog of condition (2.7). This condition (stated in [12]) is very awkward and will not be given here; we only note that it holds if $a_{ij} \equiv 0$ for $i \neq j$. We assume that $a_{ij} \in C^1(\Omega \times \mathbb{R}^3)$, $f \in C^{\alpha}(\Omega \times \mathbb{R} \times \mathbb{R}^3)$. The difficulties arising from the appearance of the coefficients $a_{ij}(x_i, x_j, \nabla u)$ can be overcome in the same way as in [12].

Note that, as is seen from Theorem 2.1, the specifics of the two-dimensional case allows us to consider equations with coefficients depending on all the variables. Moreover, the requirement of the differentiability of the leading coefficients can be replaced by the condition of their continuity in the sense of Hölder.

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