Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 235 (2007) 376–396
Journal of Differential Equations
www.elsevier.com/locate/jde

The problem of Dirichlet for anisotropic quasilinear degenerate elliptic equations

Alkis S. Tersenov [a,*], Aris S. Tersenov [b]

[a] University of Crete, Department of Mathematics, 71409 Heraklion-Crete, Greece
[b] University of Peloponnese, Department of Computer Science and Technology, 22100 Tripoli, Greece

Received 11 January 2006; revised 8 October 2006

Available online 24 January 2007

Abstract

We consider the Dirichlet problem for a class of anisotropic degenerate elliptic equations. New a priori estimates for solutions and for the gradient of solutions are established. Based on these estimates sufficient conditions guaranteeing the solvability of the problem are formulated. The results are new even in the semilinear case when the principal part is the Laplace operator.

MSC: 35J25; 35J70; 35B45

Keywords: Degenerate elliptic equations; Semilinear elliptic equations; A priori estimates

0. Introduction and main results

In the present paper we consider the following quasilinear degenerate elliptic equation

$$-\sum_{i=1}^{n} \mu_i \left(|u_{x_i}|^{p_i} u_{x_i}\right)_{x_i} = c(\mathbf{x})g(u) + f(\mathbf{x}) \quad \text{in } \Omega \subset \mathbf{R}^n, \tag{0.1}$$

coupled with homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega. \tag{0.2}$$

* Corresponding author.
E-mail addresses: tersenov@math.uoc.gr (Al.S. Tersenov), aterseno@uop.gr (Ar.S. Tersenov).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.01.009

Here constants $\mu_i > 0$ and $p_i \geqslant 0$. Without loss of generality we assume that

$$\Omega \subset \{\mathbf{x}\colon -l_i \leqslant x_i \leqslant l_i,\ i = 1, \dots, n\}.$$

Concerning the function g we suppose that

$$g(0) = 0, \qquad g(z) > 0 \quad \text{if } z > 0 \quad \text{and} \quad \left|g(z)\right| \leqslant g(C) \quad \text{for } |z| \leqslant C, \tag{0.3}$$

where C is an arbitrary positive constant. For example, functions $g(u) = \ln(|u| + 1)$ and $g(u) = u^q$ with arbitrary $q \geqslant 0$ satisfy (0.3). In the case that $g(u) = u^q$ is defined only for $u \geqslant 0$ one can take its odd or even continuation of the form $g(u) = |u|^{q-1}u$, $g(u) = |u|^q$. On the other hand, it is obvious that (0.3) does not restrict us only to odd or even functions. For example $g(u) = e^u - 1$, which is neither odd nor even, satisfies condition (0.3).

For the equation

$$-\sum_{i=1}^{n} \mu_i \left(|u_{x_i}|^{p_i} u_{x_i}\right)_{x_i} = f(\mathbf{x}) \quad \text{in } \Omega \subset \mathbf{R}^n$$

with condition (0.2) the existence of the unique generalized solution follows from [8]. In [8] the initial boundary value problem for the related parabolic equation was considered, but the method proposed there can be easily applied to the elliptic case. From [8] it follows that if $f \in W^{-1,p_0}(\Omega)$ (where $p_0 = \max_{1 \leqslant i \leqslant n}\{p_i'\}$, $1/p_i + 1/p_i' = 1$), then there exists a unique generalized solution such that $u \in L_2(\Omega)$ and $u_{x_i} \in L_{p_i}(\Omega)$.

The existence and nonexistence of positive solutions for equation

$$-\sum_{i=1}^{n} \mu_i \left(|u_{x_i}|^{p_i} u_{x_i}\right)_{x_i} = \lambda u^q, \quad \lambda > 0 \text{ is constant},$$

coupled with boundary condition (0.2) was considered in [3]. The existence result was proved in the subcritical case and nonexistence result in the at least critical case.

In [7] the regularity question for the equation

$$\sum_{i=1}^{n} \mu_i \left(|u_{x_i}|^{p_i} u_{x_i}\right)_{x_i} = 0,$$

was considered. It was proved that each component of the gradient is bounded in L_∞ norm under the assumption that the solution is bounded.

Our goal is to obtain sufficient conditions for the solvability of problem (0.1), (0.2). The results obtained in the present paper are new even for the semilinear case, i.e. for $p_1 = p_2 = \cdots = p_n = 0$.

Assume that there exists a positive constant M such that

$$\left(c_0 g(M) + f_0\right)\left(\frac{3l^2 + 2l}{2}\right)^{p+1} < \mu(p+1)M^{p+1}. \tag{0.4}$$

Here $p = p_{i_0} = \max\{p_1, \dots, p_n\}$, $\mu = \mu_{i_0}$, $l = l_{i_0}$, $c_0 = \sup_\Omega |c(\mathbf{x})|$ and $f_0 = \sup_\Omega |f(\mathbf{x})|$.

Below we will give several examples concerning this condition.

Remark 1. Instead of (0.4) we can take one of the following two assumptions.

1. Suppose that there exists a positive constant M such that

$$\big(c_0 g(M)+f_0\big)\left(\frac{3\tilde{l}^2+2\tilde{l}}{2}\right)^{\tilde{p}+1} < \tilde{\mu}(\tilde{p}+1)M^{\tilde{p}+1}. \tag{0.4$_1$}$$

Here $\tilde{l}=l_{i_1}=\min\{l_1,\dots,l_n\}$, $\tilde{\mu}=\mu_{i_1}$, $\tilde{p}=p_{i_1}$.

2. Suppose that there exists a positive constant M such that

$$\big(c_0 g(M)+f_0\big)\left(\frac{3\hat{l}^2+2\hat{l}}{2}\right)^{\hat{p}+1} < \hat{\mu}(\hat{p}+1)M^{\hat{p}+1}. \tag{0.4$_2$}$$

Here $\hat{\mu}=\mu_{i_2}=\max\{\mu_1,\dots,\mu_n\}$, $\hat{l}=l_{i_2}$, $\hat{p}=p_{i_2}$.

Definition 1. We say that function $u(\mathbf{x})$ is a generalized solution of problem (0.1), (0.2) if $u(\mathbf{x})\in W^{1,\infty}(\Omega)$, $u(\mathbf{x})=0$ for $\mathbf{x}\in\partial\Omega$ and

$$\int_\Omega \sum_{i=1}^n \mu_i |u_{x_i}|^{p_i} u_{x_i}\phi_{x_i}(\mathbf{x})\,d\mathbf{x} = \int_\Omega \big(c(\mathbf{x})g(u)+f(\mathbf{x})\big)\phi(\mathbf{x})\,d\mathbf{x} \quad \forall\phi\in \overset{\circ}{W}{}^{1,r}(\Omega),\ 1\leqslant r<+\infty.$$

Theorem 1.

(i) *Suppose that $c(\mathbf{x})$ and $f(\mathbf{x})$ are bounded in $\bar{\Omega}$, $g(u)$ is a Hölder continuous function on $[-M,M]$ and conditions* (0.3), (0.4) *are fulfilled. If the domain $\Omega\subset\mathbf{R}^n$ is strictly convex, then there exists a generalized solution of problem* (0.1), (0.2) *such that*

$$\|u\|_{L_\infty(\Omega)}\leqslant M_0 \quad \text{and} \quad \|u_{x_i}\|_{L_\infty(\Omega)}\leqslant (1+2l_i)\left(\frac{\Phi_0}{\mu_i(1+p_i)}\right)^{\frac{1}{p_i+1}}, \quad i=1,\dots,n,$$

where $M_0=\inf\{M\colon M$ satisfies (0.4)$\}$ *and*

$$\Phi_0=\max_{\bar{\Omega}\times[-M_0,M_0]}\big|c(\mathbf{x})g(u)+f(\mathbf{x})\big|.$$

(ii) *If in addition $c(\mathbf{x})\leqslant 0$ and $g(u)$ is a nondecreasing function then the solution is unique.*

Example 1. Consider the following equation

$$-\sum_{i=1}^n \mu_i\big(|u_{x_i}|u_{x_i}\big)_{x_i} = c(\mathbf{x})u^4+f(\mathbf{x}) \quad \text{in } \Omega. \tag{0.5}$$

Here $p=1$ and $g(u)=u^4$. Let $c_0>0$, then inequality (0.4) takes the following form

$$c_0M^4-\frac{8\mu}{(3l^2+2l)^2}M^2+f_0<0$$

or for $\bar{M} = M^2$

$$c_0\bar{M}^2 - \frac{8\mu}{(3l^2+2l)^2}\bar{M} + f_0 < 0. \tag{0.6}$$

Obviously, $\bar{M} > 0$ satisfying (0.6) exists if

$$f_0 \leqslant \frac{16\mu^2}{c_0(3l^2+2l)^4}. \tag{0.7}$$

Thus if the function $f(\mathbf{x})$ satisfies condition (0.7), then Theorem 1 guarantees the existence of a generalized solution of problem (0.5), (0.2) satisfying inequalities

$$\|u\|_{L_\infty(\Omega)} \leqslant M_0 \quad \text{and} \quad \|u_{x_i}\|_{L_\infty(\Omega)} \leqslant (1+2l_i)\left(\frac{c_0M_0^4 + f_0}{2\mu_i}\right)^{\frac{1}{2}}, \quad i = 1, \ldots, n,$$

with

$$M_0 = \left(\frac{4\mu}{c_0(3l^2+2l)^2} - \left(\frac{16\mu^2}{c_0^2(3l^2+2l)^4} - \frac{f_0}{c_0}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Example 2. If $g(u) = u^q$ (or $g(u) = |u|^{q-1}u$ or $g(u) = |u|^q$) and $p+1 > q$ then for arbitrary bounded $f(\mathbf{x})$ one can find a positive M satisfying condition (0.4) and as a consequence obtain the existence of a generalized solution by Theorem 1.

Example 3. If $c_0 = 0$, then (as in the previous case) one can always find positive M satisfying (0.4) and obtain the existence of a generalized solution for any bounded $f(\mathbf{x})$. In this case

$$\|u\|_{L_\infty(\Omega)} \leqslant M_0 = \frac{3l^2+2l}{2}\left(\frac{f_0}{\mu(p+1)}\right)^{\frac{1}{p+1}},$$

$$\|u_{x_i}\|_{L_\infty(\Omega)} \leqslant (1+2l_i)\left(\frac{f_0}{\mu_i(1+p_i)}\right)^{\frac{1}{p_i+1}}, \quad i = 1, \ldots, n.$$

Consider now the semilinear equation ($p_i = 0$ for all i). For simplicity suppose that $\mu_i = \mu$ for all i:

$$-\mu\Delta u = c(\mathbf{x})g(u) + f(\mathbf{x}) \quad \text{in } \Omega \subset \mathbf{R}^n. \tag{0.8}$$

In this case the use of (0.4_1) is appropriate, for $p_i = 0$ it takes the form

$$\big(c_0 g(M) + f_0\big)\frac{3\tilde{l}^2 + 2\tilde{l}}{2} < \mu M. \tag{0.9}$$

In [9] it was shown that problem (0.8), (0.2) with $f \equiv 0$ and $c(\mathbf{x}) \equiv \text{const}$ has a nontrivial solution. It is natural to expect that problem (0.8), (0.2) with arbitrary f may have no solution. Let us mention here that many papers have recently appeared (see [1,2,10] and the references there) where for the problem

$$-\Delta u = |u|^{q-1}u + f(\mathbf{x}) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

different sufficient conditions on existence are formulated.

Theorem 2.

(i) *Suppose that* $c(\mathbf{x}), f(\mathbf{x}) \in C^{\gamma}(\bar{\Omega})$, $g(u) \in C^{\gamma}([-M, M])$, $\partial\Omega \in C^{2+\gamma}$, $\gamma \in (0,1)$. *If* (0.9) *is fulfilled then there exists a classical solution of problem* (0.8), (0.2) $u(\mathbf{x}) \in C^{2+\gamma}(\bar{\Omega})$ *such that*

$$\max_{\Omega} |u| \leqslant M_0,$$

where $M_0 = \inf\{M\colon\ M \text{ satisfies } (0.9)\}$.

(ii) *If in addition* $c(\mathbf{x}) \leqslant 0$ *and* $g(u)$ *is a nondecreasing function then the solution is unique.*

Remark 2. In order to prove Theorem 1 we need to obtain a priori estimates for u and ∇u. To prove the estimate for ∇u we will use the convexity of the domain. In order to prove Theorem 2 we need to obtain a priori estimates only for u where we do not need the convexity of the domain. If the domain in Theorem 2 is strictly convex then we additionally have

$$\max_{\Omega} |u_{x_i}| \leqslant (1 + 2l_i)\frac{\Phi_0}{\mu}, \quad i = 1, \ldots, n.$$

Example 4. Consider the equation

$$-\mu\Delta u = \lambda u^2 + f(\mathbf{x}), \quad \text{where } \lambda \text{ is constant.} \tag{0.10}$$

Condition (0.9) takes the form

$$|\lambda|M^2 - \frac{2\mu}{3\tilde{l}^2 + 2\tilde{l}}M + f_0 < 0.$$

From Theorem 2 we have that there exists a classical solution if

$$f_0 \leqslant \frac{\mu^2}{|\lambda|(3\tilde{l}^2 + 2\tilde{l})^2},$$

and in this case

$$\max_{\Omega} |u| \leqslant M_0 = \frac{\mu}{|\lambda|(3\tilde{l}^2 + 2\tilde{l})} - \left(\frac{\mu^2}{\lambda^2(3\tilde{l}^2 + 2\tilde{l})^2} - \frac{f_0}{|\lambda|}\right)^{\frac{1}{2}}.$$

If the domain is strictly convex then in addition

$$\max_{\Omega} |u_{x_i}| \leqslant (1 + 2l_i)\frac{|\lambda|M_0^2 + f_0}{\mu}, \quad i = 1, \ldots, n.$$

Example 5. Consider

$$-\mu\Delta u=\lambda e^{u}. \tag{0.11}$$

Here $g(u)=e^{u}-1$ and $f=\lambda$. Condition (0.9) takes the form: there exists $M>0$ such that

$$e^{M}-CM<0,$$

where

$$C=\frac{2\mu}{|\lambda|(3\tilde{l}^{2}+2\tilde{l})}.$$

This fact is equivalent to say that the minimum of the function $e^{x}-Cx$ must be negative, that is

$$\frac{2\mu}{|\lambda|(3\tilde{l}^{2}+2\tilde{l})}>e. \tag{0.12}$$

From Theorem 2 we have that there exists a classical solution of (0.11), (0.2) if (0.12) is fulfilled and for this solution we have

$$\max_{\Omega}|u|\leqslant M_{0}=\ln C. \tag{0.13}$$

If the domain is strictly convex then in addition

$$\max_{\Omega}|u_{x_i}|\leqslant(1+2l_i)\frac{|\lambda|e^{\ln C}}{\mu}=\frac{2(1+2l_i)}{3\tilde{l}^{2}+2\tilde{l}},\quad i=1,\dots,n. \tag{0.14}$$

Recall that in [4] for the problem

$$-\Delta u=e^{u}\quad\text{in }\Omega\subset\mathbf{R}^{n},\qquad u|_{\partial\Omega}=0 \tag{0.15}$$

it was shown that there exists a positive number κ depending on the dimension n such that if the diameter of Ω is less than κ there exists (at least one) solution of problem (0.15). From Example 5 it follows that there exists (at least one) solution of problem (0.15) if the size of the domain Ω at least in one direction is small enough (independently of the dimension) namely

$$3\tilde{l}^{2}+2\tilde{l}<\frac{2}{e}.$$

Finally let us mention that similarly to problem (0.11), (0.2) we can show that if (0.12) is fulfilled then there exists a classical solution of the problem

$$\mu\Delta u=\lambda e^{u}\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0$$

satisfying inequality (0.13) (and, if Ω is strictly convex, inequality (0.14)). Moreover from Theorem 2 it follows that the solution is unique.

The paper consists of two sections. In the first section we obtain the a priori estimate for the regularized problem and in the second based on these a priori estimates we prove Theorems 1, 2.

1. A priori estimates for the regularized problem

Let us start from problem (0.1), (0.2). Consider the regularized equation

$$-\sum_{i=1}^{n}\mu_i\left(\left(u_{x_i}^{\alpha}+\varepsilon\right)^{p_i/\alpha}u_{x_i}\right)_{x_i}=c_\varepsilon(\mathbf{x})g_M(u)+f_\varepsilon(\mathbf{x}). \tag{1.1}$$

Here the constant $\alpha\in(0,1)$ is such that $(u_{x_i}^{\alpha})^{p_i/\alpha}=|u_{x_i}|^{p_i}$ (for example $\alpha=2/3$); constant $\varepsilon>0$, $c_\varepsilon(\mathbf{x})$, $f_\varepsilon(\mathbf{x})$ are Hölder continuous functions such that

$$c_\varepsilon(\mathbf{x})\to c(\mathbf{x})\quad\text{and}\quad f_\varepsilon(\mathbf{x})\to f(\mathbf{x})\quad\text{in } L_\infty \text{ norm, when } \varepsilon\to 0,$$

where without loss of generality we assume that

$$\max\left|c_\varepsilon(\mathbf{x})\right|=c_0,\qquad \max\left|f_\varepsilon(\mathbf{x})\right|=f_0,\qquad \max\left|c_\varepsilon(\mathbf{x})g(u)+f_\varepsilon\right|=\Phi_0;$$

and finally

$$g_M(z)=\begin{cases} g(z), & \text{for } |z|\leqslant M,\\ g(M), & \text{for } z>M,\\ g(-M), & \text{for } z<-M.\end{cases} \tag{1.2}$$

Obviously from (0.3) we have $-g(M)\leqslant g_M(u)\leqslant g(M)$.

The first step is to obtain the estimate $|u|\leqslant M$ for a solution of problem (1.1), (0.2). After this in (1.1) instead of $g_M(u)$ we can take $g(u)$ (due to (1.2)).

Lemma 1. *If* (0.3) *and* (0.4) *are fulfilled, then for any classical solution of problem* (1.1), (0.2) *the following estimate is valid*

$$\left|u(\mathbf{x})\right|\leqslant M.$$

Proof. Without loss of generality we assume that $i_0=1$, i.e. $p=p_1$, $l=l_1$, $\mu=\mu_1$. Rewrite Eq. (1.1) in nondivergent form

$$-\sum_{i=1}^{n}a_{i\varepsilon}(u_{x_i})u_{x_ix_i}=c_\varepsilon(\mathbf{x})g_M(u)+f_\varepsilon(\mathbf{x}), \tag{1.3}$$

where

$$a_{i\varepsilon}(z)=\mu_i\left(z^\alpha+\varepsilon\right)^{\frac{p_i}{\alpha}-1}\left((p_i+1)z^\alpha+\varepsilon\right).$$

Define the function $h(x_1)$:

$$h(x_1)=\tilde{M}\left(\frac{l^2-x_1^2}{2}+(1+l)(l+x_1)\right),$$

where

$$\tilde{M} = \frac{2M}{3l^2 + 2l}.$$

Obviously for

$$Lu \equiv -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})u_{x_i x_i}$$

we have

$$Lu = c_\varepsilon(\mathbf{x})g_M(u) + f_\varepsilon(\mathbf{x}) \tag{1.4}$$

and

$$\begin{aligned} Lh &= -\sum_{i=1}^{n} a_{i\varepsilon}(h_{x_i})h_{x_i x_i} = -a_{1\varepsilon}(h_{x_1})h_{x_1 x_1} \\ &= \mu\big(h'^{\alpha} + \varepsilon\big)^{\frac{p}{\alpha}-1}\big((p+1)h'^{\alpha} + \varepsilon\big)\tilde{M} \geqslant \mu(p+1)\tilde{M}^{p+1}. \end{aligned} \tag{1.5}$$

Here we use the fact that $h'(x_1) \geqslant \tilde{M}$, $h''(x_1) = -\tilde{M}$ and $\alpha < 1$. One can easily see that if $\alpha \in (0, 1)$ then for any $p \geqslant 0$ the expression $\mu(h'^{\alpha} + \varepsilon)^{\frac{p}{\alpha}-1}((p+1)h'^{\alpha} + \varepsilon)\tilde{M}$ is a nondecreasing with respect to ε function. For the function

$$v(\mathbf{x}) \equiv u(\mathbf{x}) - h(x_1)$$

we have

$$\begin{aligned} Lu - Lh &= -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})u_{x_i x_i} + \sum_{i=1}^{n} a_{i\varepsilon}(h_{x_i})h_{x_i x_i} \\ &= -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})v_{x_i x_i} + \big(a_{1\varepsilon}(h_{x_1}) - a_{1\varepsilon}(u_{x_1})\big)h_{x_1 x_1}. \end{aligned}$$

On the other hand, due to (1.4), (1.5) we have

$$Lu - Lh = c_\varepsilon(\mathbf{x})g_M(u) + f_\varepsilon(\mathbf{x}) - Lh \leqslant c_\varepsilon(\mathbf{x})g_M(u) + f_\varepsilon(\mathbf{x}) - \mu(p+1)\tilde{M}^{p+1}.$$

Hence

$$\begin{aligned} -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})v_{x_i x_i} &\leqslant \big(a_{1\varepsilon}(u_{x_1}) - a_{1\varepsilon}(h_{x_1})\big)h_{x_1 x_1} \\ &\quad + c_\varepsilon(\mathbf{x})g_M(u) + f_\varepsilon(\mathbf{x}) - \mu(p+1)\tilde{M}^{p+1}. \end{aligned} \tag{1.6}$$

Suppose that at the point $N \in \bar{\Omega} \setminus \partial\Omega$ the function $v(\mathbf{x})$ attains its positive maximum. At this point we have $v > 0$ and $v_{x_i} = 0$ or $u > h \geqslant 0$ and $u_{x_1} = h' \geqslant \tilde{M}$, $u_{x_i} = 0$ for $i = 2, \ldots, n$ (in particular $a_{1\varepsilon}(u_{x_1}) - a_{1\varepsilon}(h_{x_1}) = 0$). Thus

$$\begin{aligned}
-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) v_{x_i x_i}\Big|_N &\leqslant c_\varepsilon(\mathbf{x}) g_M(u) + f_\varepsilon(\mathbf{x}) - \mu(p+1)\tilde{M}^{p+1}\big|_N \\
&\leqslant \max\big|c_\varepsilon(\mathbf{x})\big| g(M) + \max\big|f_\varepsilon(\mathbf{x})\big| - \mu(p+1)\tilde{M}^{p+1} \\
&= c_0 g(M) + f_0 - \mu(p+1)\left(\frac{2M}{3l^2 + 2l}\right)^{p+1}. \qquad (1.7)
\end{aligned}$$

Here we use the fact that for positive u we have $0 \leqslant g_M(u) \leqslant g(M)$. Hence due to (0.4)

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) v_{x_i x_i}\Big|_N < 0.$$

This contradicts the assumption that $v(\mathbf{x})$ attains its positive maximum at N. Due to the homogeneous boundary conditions, on $\partial\Omega$ we have $v = -h \leqslant 0$. Taking into account that $v(\mathbf{x})$ cannot attain its positive maximum in $\bar{\Omega} \setminus \partial\Omega$ we conclude that

$$v(\mathbf{x}) \leqslant 0 \quad \text{or} \quad u(\mathbf{x}) \leqslant h(x_1) \quad \text{in } \bar{\Omega}.$$

Now let us obtain the estimate from the below.

For the function $w(\mathbf{x}) \equiv u(\mathbf{x}) + h(x_1)$ we have

$$\begin{aligned}
Lu + Lh &= -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) u_{x_i x_i} - \sum_{i=1}^{n} a_{i\varepsilon}(h_{x_i}) h_{x_i x_i} \\
&= -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i} - \big(a_{1\varepsilon}(h_{x_1}) - a_{1\varepsilon}(u_{x_1})\big) h_{x_1 x_1}.
\end{aligned}$$

On the other hand,

$$Lu + Lh = c_\varepsilon(\mathbf{x}) g_M(u) + f_\varepsilon(\mathbf{x}) + Lh \geqslant c_\varepsilon(\mathbf{x}) g_M(u) + f_\varepsilon(\mathbf{x}) + \mu(p+1)\tilde{M}^{p+1}.$$

Thus

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i} \geqslant \big(a_{1\varepsilon}(h_{x_1}) - a_{1\varepsilon}(u_{x_1})\big) h_{x_1 x_1} + c_\varepsilon(\mathbf{x}) g_M(u) + f_\varepsilon(\mathbf{x}) + \mu(p+1)\tilde{M}^{p+1}.$$

Suppose that at the point $N_1 \in \bar{\Omega} \setminus \partial\Omega$ the function $w(\mathbf{x})$ attains its negative minimum. At this point we have $w < 0$ and $w_{x_i} = 0$ or $u < -h \leqslant 0$ and $u_{x_1} = -h_{x_1} \leqslant -\tilde{M}$, $u_{x_i} = 0$ for $i = 2, \ldots, n$. Therefore (due to the fact that $a_{i\varepsilon}(z) = a_{i\varepsilon}(-z)$)

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i}\Big|_{N_1} \geqslant c_\varepsilon(\mathbf{x}) g_M(u) + f_\varepsilon(\mathbf{x}) + \mu(p+1)\tilde{M}^{p+1}\big|_{N_1}$$

$$\geqslant -c_0 g(M) - f_0 + \mu(p+1)\left(\frac{2M}{3l^2+2l}\right)^{p+1}. \tag{1.8}$$

Here we use the inequality

$$c_\varepsilon(N_1) g_M\big(u(N_1)\big) \geqslant -c_0 g(M).$$

If $c_\varepsilon(N_1) \geqslant 0$, then the last inequality follows from the fact that $g_M(u) \geqslant -g(M)$. If $c_\varepsilon(N_1) < 0$, then the inequality follows from the fact that $g_M(u) \leqslant g(M)$. Hence from (1.8) (due to (0.4)) we obtain

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i}\Big|_{N_1} > 0.$$

This contradicts the assumption that $w(\mathbf{x})$ attains its (negative) minimum at N_1.

Due to the homogeneous boundary condition, on $\partial\Omega$ we have $w = h \geqslant 0$. Taking into account that $w(\mathbf{x})$ cannot attain its negative minimum in $\bar{\Omega} \setminus \partial\Omega$ we conclude that

$$w(\mathbf{x}) \geqslant 0 \quad \text{or} \quad u(\mathbf{x}) \geqslant -h(x_1) \quad \text{in } \bar{\Omega}.$$

Consequently

$$-h(x_1) \leqslant u(\mathbf{x}) \leqslant h(x_1). \tag{1.9}$$

Now taking $\tilde{h}(x_1) \equiv h(-x_1)$ instead of $h(x_1)$ we obtain

$$-\tilde{h}(x_1) \leqslant u(\mathbf{x}) \leqslant \tilde{h}(x_1). \tag{1.10}$$

This estimate can be easily established in the same way as (1.9) because $\tilde{h}'^{\alpha} \geqslant \tilde{M}^{\alpha}$ and $\tilde{h}'' = -\tilde{M}$. The first inequality ($\tilde{h}'^{\alpha} \geqslant \tilde{M}^{\alpha}$) follows from $-\tilde{h}' \geqslant \tilde{M} \geqslant 0$ due to the choice of α ($\alpha = 2/3$).

From (1.9) and (1.10) we conclude that

$$\big|u(\mathbf{x})\big| \leqslant h(0) = \tilde{h}(0) = \frac{3l^2+2l}{2}\tilde{M} = M. \qquad \square$$

Remark 3. Represent M for a sufficiently small ϵ as $M = M_0 + \epsilon$, where $M_0 = \inf\{M$: M satisfies (0.4)$\}$. Passing to the limit when $\epsilon \to 0$ we obtain the estimate

$$\big|u(\mathbf{x})\big| \leqslant M_0.$$

Let us turn to the estimate of the derivatives. First in Lemma 2 we will obtain the auxiliary result which actually is the boundary gradient estimate. Then in Lemma 3 we will obtain the global gradient estimate. We additionally suppose now that Ω is strictly convex and that the parts of $\partial\Omega$ lying in the half spaces $x_i \leqslant 0$ and $x_i \geqslant 0$ can be expressed as

$$x_i = F_i \quad \text{and} \quad x_i = G_i, \quad i = 1, \dots, n,$$

respectively. Here the functions F_i and G_i depend on all variables except of x_i. Due to the convexity we have

$$F_{kx_ix_i} \geqslant 0, \quad G_{kx_ix_i} \leqslant 0, \quad k = 1, 2, \dots, n, \; i = 1, 2, \dots, n.$$

Define the function $\mathrm{h}_k(\tau)$ by the following

$$\mathrm{h}_k(\tau) = -C_k \frac{\tau^2}{2} + \big[C_k(1 + 2l_k) + \epsilon\big]\tau, \quad \tau \in [0, 2l_k], \tag{1.11}$$

where

$$C_k = \left(\frac{\Phi_0}{\mu_k(p_k + 1)}\right)^{\frac{1}{p_k+1}}.$$

Recall that $\Phi_0 = \max_{\bar{\Omega}\times[-M,M]} |c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x})|$. Obviously

$$\mathrm{h}_k'' = -C_k, \qquad \mathrm{h}_k' \geqslant C_k + \epsilon > C_k.$$

Lemma 2. *If conditions* (0.3), (0.4) *are fulfilled and Ω is strictly convex then for any classical solution of problem* (1.1), (0.2) *the following estimates are valid*

$$\big|u(\mathbf{x})\big| \leqslant \mathrm{h}_k(G_k - x_k), \quad \big|u(\mathbf{x})\big| \leqslant \mathrm{h}_k(x_k - F_k), \quad k = 1, \dots, n, \; \textit{in } \bar{\Omega}.$$

Proof. We will prove these estimates for $k = 1$, the other cases can be considered similarly. Let us start from the first inequality. Introduce the function

$$v(\mathbf{x}) \equiv u(\mathbf{x}) - \mathrm{h}_1(\zeta), \quad \text{where } \zeta = G_1(x_2, x_3, \dots, x_n) - x_1.$$

We have

$$Lu(\mathbf{x}) \equiv -\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})u_{x_ix_i} = c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}),$$

$$L\mathrm{h}_1(\zeta) = -\sum_{i=1}^{n} a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\zeta)\big)\mathrm{h}_{1x_ix_i}(\zeta)$$

$$= -a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big)\mathrm{h}_1''(\zeta) - \sum_{i=2}^{n} a_{i\varepsilon}\big(\mathrm{h}_1'(\zeta)G_{1x_i}\big)\big(\mathrm{h}_1''(\zeta)G_{1x_i}^2 + \mathrm{h}_1'(\zeta)G_{1x_ix_i}\big).$$

Due to the convexity of Ω we have $G_{1x_ix_i} \leqslant 0$ and taking into account that $\mathrm{h}_1'' \leqslant 0$, $\mathrm{h}_1' \geqslant 0$ we conclude that

$$\mathrm{h}_1''(\zeta)G_{1x_i}^2 + \mathrm{h}_1'(\zeta)G_{1x_ix_i} \leqslant 0$$

and hence

$$L\mathrm{h}_1(\zeta) \geqslant -a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big)\mathrm{h}_1''(\zeta) = C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big). \tag{1.12}$$

Thus

$$Lu(\mathbf{x}) - L\mathrm{h}_1(\zeta) \leqslant c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}) - C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big). \tag{1.13}$$

On the other hand,

$$Lu(\mathbf{x}) - L\mathrm{h}_1(\zeta) = -\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})v_{x_ix_i} - \sum_{i=1}^n \big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\zeta)\big)\big]\mathrm{h}_{1x_ix_i}(\zeta). \tag{1.14}$$

Hence, from (1.13), (1.14) we obtain

$$\begin{aligned} -\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})v_{x_ix_i} \leqslant{} & \sum_{i=1}^n \big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\zeta)\big)\big]\mathrm{h}_{1x_ix_i}(\zeta) \\ & + c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}) - C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big). \end{aligned} \tag{1.15}$$

Suppose that at the point $N \in \bar{\Omega} \setminus \partial\Omega$ the function $v(\mathbf{x})$ attains its maximum. At this point we have $v_{x_i} = 0$ or $u_{x_i}(\mathbf{x}) = \mathrm{h}_{1x_i}(\zeta)$ (in particular $a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}(\mathrm{h}_{1x_i}(\zeta)) = 0$, $i = 1, \dots, n$) and hence

$$\begin{aligned} -\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})v_{x_ix_i}\Big|_N &\leqslant c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}) - C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big)\big|_N \\ &< c_\varepsilon(N)g\big(u(N)\big) + f_\varepsilon(N) - \mu_1(p_1+1)C_1^{p_1+1} \leqslant 0. \end{aligned}$$

This contradicts the assumption that $v(\mathbf{x})$ attains its maximum at the internal point of the domain Ω. Due to the fact that $v = -\mathrm{h}_1 \leqslant 0$ on $\partial\Omega$ we conclude that

$$v(\mathbf{x}) \leqslant 0 \quad \text{or} \quad u(\mathbf{x}) \leqslant \mathrm{h}_1(G_1 - x_1) \quad \text{in } \bar{\Omega}.$$

Next we obtain a lower bound. Introduce the function $w(\mathbf{x}) \equiv u(\mathbf{x}) + \mathrm{h}_1(\zeta)$. Similarly to (1.13) and (1.14) we obtain

$$Lu(\mathbf{x}) + L\mathrm{h}_1(\zeta) \geqslant c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}) + C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big)$$

and

$$Lu(\mathbf{x}) + L\mathrm{h}_1(\zeta) = -\sum_{i=1}^n a_{i\varepsilon}(u_{x_i})w_{x_ix_i} + \sum_{i=1}^n \big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\zeta)\big)\big]\mathrm{h}_{1x_ix_i}(\zeta).$$

Hence

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i} \geqslant -\sum_{i=1}^{n} \big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\zeta)\big)\big] \mathrm{h}_{1x_i x_i}(\zeta) + c_\varepsilon(\mathbf{x}) g(u) + f_\varepsilon(\mathbf{x}) + C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\zeta)\big). \tag{1.16}$$

Suppose that at the point $N_1 \in \bar{\Omega} \setminus \partial\Omega$ the function $w(\mathbf{x})$ attains its minimum. At this point we have $w_{x_i} = 0$ or $u_{x_i}(\mathbf{x}) = \mathrm{h}_{1x_i}(\zeta)$ (in particular $a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}(h_{1x_i}(\zeta)) = 0$, $i = 1, \ldots, n$) and hence

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) w_{x_i x_i}\Big|_{N_1} \geqslant c_\varepsilon(\mathbf{x}) g(u) + f_\varepsilon(\mathbf{x}) + C_1 a_{1\varepsilon}\big(\mathrm{h}'(\zeta)\big)\big|_{N_1} > c_\varepsilon(N) g\big(u(N)\big) + f_\varepsilon(N) + \mu_1(p_1+1) C_1^{p_1+1} \geqslant 0.$$

This contradicts the assumption that $w(\mathbf{x})$ attains its minimum at the internal point of the domain Ω. Due to the fact that $w = \mathrm{h}_1 \geqslant 0$ on $\partial\Omega$ we conclude that

$$w(\mathbf{x}) \geqslant 0 \quad \text{or} \quad u(\mathbf{x}) \geqslant -\mathrm{h}_1(G_1 - x_1) \quad \text{in } \bar{\Omega}.$$

Thus the estimate $|u(\mathbf{x})| \leqslant \mathrm{h}_1(G_1 - x_1)$ in $\bar{\Omega}$ is proved.

Now introduce functions $\tilde{v}(\mathbf{x}) \equiv u(\mathbf{x}) - \mathrm{h}_1(\eta)$ and $\tilde{w}(\mathbf{x}) \equiv u(\mathbf{x}) + \mathrm{h}_1(\eta)$ where $\eta = x_1 - F_1(x_2, x_3, \ldots, x_n)$. Similarly to (1.12) we obtain

$$L\mathrm{h}_1(\eta) = -\sum_{i=1}^{n} a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\eta)\big) \mathrm{h}_{1x_i x_i}(\eta) = -a_{1\varepsilon}\big(\mathrm{h}_1'(\eta)\big)\mathrm{h}_1''(\eta) - \sum_{i=2}^{n} a_{i\varepsilon}\big(\mathrm{h}_1'(\eta) F_{1x_i}\big)\big(\mathrm{h}_1''(\eta) F_{1x_i}^2 - \mathrm{h}_1'(\eta) F_{1x_i x_i}\big).$$

Due to the convexity of Ω we have $F_{1x_i x_i} \geqslant 0$ and taking into account that $\mathrm{h}_1'' \leqslant 0$, $\mathrm{h}_1' \geqslant 0$ we conclude that

$$\mathrm{h}_1''(\zeta) F_{1x_i}^2 - \mathrm{h}_1'(\zeta) F_{1x_i x_i} \leqslant 0$$

and hence

$$L\mathrm{h}_1(\eta) \geqslant -a_{1\varepsilon}\big(\mathrm{h}_1'(\eta)\big)\mathrm{h}_1''(\eta) = C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\eta)\big).$$

Furthermore similarly to (1.15) and (1.16) we obtain

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i}) \tilde{v}_{x_i x_i} \leqslant \sum_{i=1}^{n} \big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\eta)\big)\big] \mathrm{h}_{1x_i x_i}(\eta) + c_\varepsilon(\mathbf{x}) g(u) + f_\varepsilon(\mathbf{x}) - C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\eta)\big)$$

and

$$-\sum_{i=1}^{n} a_{i\varepsilon}(u_{x_i})\tilde{w}_{x_i x_i} \geqslant -\sum_{i=1}^{n}\big[a_{i\varepsilon}(u_{x_i}) - a_{i\varepsilon}\big(\mathrm{h}_{1x_i}(\eta)\big)\big]\mathrm{h}_{1x_i x_i}(\eta)$$
$$+ c_\varepsilon(\mathbf{x})g(u) + f_\varepsilon(\mathbf{x}) + C_1 a_{1\varepsilon}\big(\mathrm{h}_1'(\eta)\big).$$

Now in the same manner as in the previous case we obtain the estimate $|u(\mathbf{x})| \leqslant \mathrm{h}_1(x_1 - F_1)$ in $\bar{\Omega}$.

Lemma is proved. □

Lemma 3. *If conditions* (0.3), (0.4) *are fulfilled and* Ω *is strictly convex, then for any classical solution of problem* (1.1), (0.2) *the following estimates are valid*

$$\big|u_{x_i}(\mathbf{x})\big| \leqslant (1 + 2l_i)\left(\frac{\Phi_0}{\mu_i(1 + p_i)}\right)^{\frac{1}{p_i+1}}, \quad i = 1, \dots, n.$$

Proof. We will prove the estimate for $i = 1$, for $i = 2, \dots, n$ the proof is similar. Consider the equations

$$-a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big)u_{x_1x_1}(\mathbf{x}) - \sum_{i=2}^{n} a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big)u_{x_ix_i}(\mathbf{x}) = c_\varepsilon(\mathbf{x})g\big(u(\mathbf{x})\big) + f_\varepsilon(\mathbf{x}), \tag{1.17}$$

$$-a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)u_{\xi\xi}(\tilde{\mathbf{x}}) - \sum_{i=2}^{n} a_{i\varepsilon}\big(u_{x_i}(\tilde{\mathbf{x}})\big)u_{x_ix_i}(\tilde{\mathbf{x}}) = c_\varepsilon(\tilde{\mathbf{x}})g\big(u(\tilde{\mathbf{x}})\big) + f_\varepsilon(\tilde{\mathbf{x}}), \tag{1.18}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\tilde{\mathbf{x}} = (\xi, x_2, \dots, x_n)$. Subtracting Eq. (1.18) from (1.17) for

$$v(\xi, \mathbf{x}) \equiv u(\mathbf{x}) - u(\tilde{\mathbf{x}})$$

we obtain

$$-a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big)v_{x_1x_1} - a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)v_{\xi\xi} - \sum_{i=2}^{n}\big[a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big)u_{x_ix_i}(\mathbf{x}) - a_{i\varepsilon}\big(u_{x_i}(\tilde{\mathbf{x}})\big)u_{x_ix_i}(\tilde{\mathbf{x}})\big]$$
$$= c_\varepsilon(\mathbf{x})g\big(u(\mathbf{x})\big) + f_\varepsilon(\mathbf{x}) - c_\varepsilon(\tilde{\mathbf{x}})g\big(u(\tilde{\mathbf{x}})\big) - f_\varepsilon(\tilde{\mathbf{x}}).$$

Rewrite this equation in the following form

$$-a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big)v_{x_1x_1} - a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)v_{\xi\xi} - \sum_{i=2}^{n} a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big)v_{x_ix_i}$$
$$= \sum_{i=2}^{n}\big[a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big) - a_{i\varepsilon}\big(u_{x_i}(\tilde{\mathbf{x}})\big)\big]u_{x_ix_i}(\tilde{\mathbf{x}})$$
$$+ c_\varepsilon(\mathbf{x})g\big(u(\mathbf{x})\big) + f_\varepsilon(\mathbf{x}) - c_\varepsilon(\tilde{\mathbf{x}})g\big(u(\tilde{\mathbf{x}})\big) - f_\varepsilon(\tilde{\mathbf{x}}). \tag{1.19}$$

Consider (1.19) in the domain

$$Q = \{(\xi, \mathbf{x}): \ \xi \in (F_1, G_1), \ x_1 \in (F_1, G_1), \ x_1 > \xi, \ (x_2, \dots, x_n) \in \Omega_1\},$$

where Ω_1 is a projection of Ω on the hyperplane $x_1 = 0$. For

$$w(\xi, \mathbf{x}) = v(\xi, \mathbf{x}) - \mathrm{h}_1(x_1 - \xi)$$

we have

$$\begin{aligned}
&-a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big)w_{x_1x_1} - a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)w_{\xi\xi} - \sum_{i=2}^n a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big)w_{x_ix_i} \\
&\quad = \sum_{i=2}^n \big[a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big) - a_{i\varepsilon}\big(u_{x_i}(\tilde{\mathbf{x}})\big)\big]u_{x_ix_i}(\tilde{\mathbf{x}}) \\
&\qquad + c_\varepsilon(\mathbf{x})g\big(u(\mathbf{x})\big) + f_\varepsilon(\mathbf{x}) - c_\varepsilon(\tilde{\mathbf{x}})g\big(u(\tilde{\mathbf{x}})\big) - f_\varepsilon(\tilde{\mathbf{x}}) + \big(a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big) + a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)\big)\mathrm{h}_1'' \\
&\quad \leqslant \sum_{i=2}^n \big[a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big) - a_{i\varepsilon}\big(u_{x_i}(\tilde{\mathbf{x}})\big)\big]u_{x_ix_i}(\tilde{\mathbf{x}}) + 2\Phi_0 - C_1\big(a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big) + a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)\big).
\end{aligned} \tag{1.20}$$

Suppose that at the point $N \in \bar{Q} \setminus \partial Q$ the function $w(\xi, \mathbf{x})$ attains its maximum. At this point we have $w_\xi = w_{x_i} = 0$, $i = 1, \dots, n$, or

$$u_{x_1}(\mathbf{x}) = u_\xi(\tilde{\mathbf{x}}) = \mathrm{h}_1' \quad \text{and} \quad u_{x_i}(\mathbf{x}) = u_{x_i}(\tilde{\mathbf{x}}) \quad \text{for } i = 2, \dots, n.$$

Hence from (1.20) we have

$$\begin{aligned}
&-a_{1\varepsilon}\big(u_{x_1}(\mathbf{x})\big)w_{x_1x_1} - a_{1\varepsilon}\big(u_\xi(\tilde{\mathbf{x}})\big)w_{\xi\xi} - \sum_{i=2}^n a_{i\varepsilon}\big(u_{x_i}(\mathbf{x})\big)w_{x_ix_i}\Big|_N \\
&\quad \leqslant 2\big(\Phi_0 - C_1\big(\mu_1\big(\mathrm{h}_1'\big)^{p_1}(p_1+1)\big)\big)\Big|_N < 2(\Phi_0 - \Phi_0) = 0.
\end{aligned}$$

This contradicts the assumption that $w(\xi, \mathbf{x})$ attains its maximum at the internal point of the domain Q.

Now consider $w(\xi, \mathbf{x})$ on ∂Q. The boundary of Q consists of three parts (recall that $x_1 > \xi$):

(1) $x_1 = \xi$;
(2) $\xi = F_1$, $x_1 \in [F_1, G_1]$, $x_2, \dots, x_n \in \bar{\Omega}_1$;
(3) $x_1 = G_1$, $\xi \in [F_1, G_1]$, $x_2, \dots, x_n \in \bar{\Omega}_1$.

On the first part we obviously have $w = -\mathrm{h}_1(0) = 0$. On the second and the third parts, due to Lemma 2, we have respectively

$$w = u(\mathbf{x}) - \mathrm{h}_1(x_1 - F_1) \leqslant 0$$

and

$$w = -u(\tilde{\mathbf{x}}) - \mathrm{h}_1(G_1 - \xi) \leqslant 0.$$

Consequently $w(\xi, x) \leqslant 0$ in $\bar{Q}$, which means

$$u(\mathbf{x}) - u(\tilde{\mathbf{x}}) \leqslant \mathrm{h}_1(x_1 - \xi) \quad \text{in } \bar{Q}.$$

Similarly, taking the function $\tilde{v} \equiv u(\tilde{\mathbf{x}}) - u(\mathbf{x})$ instead of v, we obtain $v \geqslant -\mathrm{h}_1(x_1 - \xi)$ in $\bar{Q}$.

By the symmetry of the variables x_1 and ξ, we consider the case $\xi > x_1$ in the same way. As a result we obtain that for $x_1 \in [F_1, G_1]$, $\xi \in [F_1, G_1]$, $(x_2, \dots, x_n) \in \Omega_1$, $|x_1 - \xi| > 0$ the following inequality holds:

$$\frac{|u(\mathbf{x}) - u(\tilde{\mathbf{x}})|}{|x_1 - \xi|} \leqslant \frac{\mathrm{h}_1(|x_1 - \xi|) - \mathrm{h}_1(0)}{|x_1 - \xi|},$$

which in turn implies the estimate $|u_{x_1}(\mathbf{x})| \leqslant \mathrm{h}_1'(0) = (1 + 2l_1)C_1 + \epsilon$. Passing to the limit when $\epsilon \to 0$ we conclude that

$$\left|u_{x_1}(\mathbf{x})\right| \leqslant (1 + 2l_1)\left(\frac{\Phi_0}{\mu_1(p_1 + 1)}\right)^{\frac{1}{p_1+1}}.$$

The lemma is proved. □

Remark 4. When proving Lemma 3 we use the idea of S.N. Kruzhkov [6] of introducing a new spatial variable for the one-dimensional quasilinear parabolic equations (see also [12] and the references there). The extension of this method to a class of multidimensional elliptic equations in convex domains was presented in [11,13]. In [11] (as well as in [6]) the right-hand side must vanish at the points where the principal part becomes zero. Of course Eq. (0.1) does not satisfy such restrictions. In [13] (as well as in [12]) it was shown that the a priori gradient estimate for the degenerated equation can be established under specific restrictions on the right-hand side which in our case look like

$$c(\mathbf{x})g(u) + f(\mathbf{x}) - c(\mathbf{x}')g(v) + f(\mathbf{x}') \leqslant 0 \quad \text{for } u > v,$$

where $x' = (x_1', x_2, x_3, \dots, x_n)$ with $x_1 > x_1'$, if we need the estimate of u_{x_1}; $x' = (x_1, x_2', x_3, \dots, x_n)$ with $x_2 > x_2'$, if we need the estimate of u_{x_2} and so on. In the present paper we have succeeded to obtain the needed estimate for (0.1) with arbitrary $c(\mathbf{x})g(u) + f(\mathbf{x})$ due to the specific form of the principal part. Note that in [6,11–13] the existence of classical solutions is proved.

Let us turn now to problem (0.8), (0.2).

Consider the regularized equation

$$-\mu\Delta u = c(\mathbf{x})g_M(u) + f(\mathbf{x}). \tag{1.21}$$

Recall that here we suppose that c and f are Hölder continuous functions. The proofs of the following two lemmas are similar to the proofs of Lemmas 1 and 3.

Lemma 4. *If* (0.3) *and* (0.4) *are fulfilled, then for any classical solution of problem* (1.21), (0.2) *the following estimate is valid*

$$\left|u(\mathbf{x})\right| \leqslant M.$$

Lemma 5. *If conditions* (0.3), (0.4) *are fulfilled and Ω is strictly convex, then for any classical solution of problem* (1.21), (0.2) *the following estimates are valid*

$$\left|u_{x_i}(\mathbf{x})\right| \leqslant (1+2l_i)\left(\frac{\Phi_0}{\mu_i(1+p_i)}\right)^{\frac{1}{p_i+1}}, \quad i=1,\dots,n.$$

2. Existence and uniqueness

Proof of Theorem 1. Consider equation

$$-\sum_{i=1}^{n} \mu_i\left(\left(u_{\varepsilon x_i}^{\alpha}+\varepsilon\right)^{p_i/\alpha} u_{\varepsilon x_i}\right)_{x_i} = c_\varepsilon(\mathbf{x}) g(u_\varepsilon) + f_\varepsilon(\mathbf{x}). \tag{2.1}$$

The classical solvability of problem (2.1), (0.2) follows from [5].

Our goal is to pass to the limit ($\varepsilon \to 0$) in (2.1) based on the a priori estimates obtained in previous section. Due to Lemmas 1 and 3 there exists a subsequence which we denote again by u_ε such that

$$u_\varepsilon(\mathbf{x}) \to u(\mathbf{x}) \quad \text{uniformly in } C_0 \text{ norm} \tag{2.2}$$

and

$$u_{\varepsilon x_i}(\mathbf{x}) \to u_{x_i}(\mathbf{x}) \quad *\,\text{weakly in } L_\infty(\Omega), \quad i=1,2,\dots,n. \tag{2.3}$$

From (2.2) it immediately follows that

$$\mathbf{f}_\varepsilon \equiv c_\varepsilon(\mathbf{x}) g(u_\varepsilon) + f_\varepsilon(\mathbf{x}) \to \mathbf{f} \equiv c(\mathbf{x}) g(u) + f(\mathbf{x}) \quad \text{strongly in } L_\infty \text{ norm}.$$

Define $A_\varepsilon(u_\varepsilon)$ and $A(u)$ elements from $W^{-1,s}(\Omega)$ (linear functionals on $\overset{\circ}{W}{}^{1,r}(\Omega)$, $\frac{1}{r}+\frac{1}{s}=1$) by the following

$$\left\langle A_\varepsilon(u_\varepsilon), v\right\rangle = \sum_{i=1}^{n} \int_{\Omega} \mu_i\left(u_{\varepsilon x_i}^{\alpha}+\varepsilon\right)^{p_i/\alpha} u_{\varepsilon x_i} v_{x_i}\, d\mathbf{x} \quad \forall v \in \overset{\circ}{W}{}^{1,r}(\Omega),$$

$$\left\langle A(u), v\right\rangle = \sum_{i=1}^{n} \int_{\Omega} \mu_i |u_{x_i}|^{p_i} u_{x_i} v_{x_i}\, d\mathbf{x} \quad \forall v \in \overset{\circ}{W}{}^{1,r}(\Omega).$$

From Lemma 3 it follows that $\mu_i(u_{\varepsilon x_i}^{\alpha}+\varepsilon)^{p_i/\alpha} u_{\varepsilon x_i}$ is bounded in $L_\infty(\Omega)$ and hence in $L_s(\Omega)$ for any s. Thus

$$A_\varepsilon(u_\varepsilon) \to \chi \quad \text{weakly in } W^{-1,s}(\Omega).$$

Our goal is to prove that

$$\chi = A(u).$$

One can easily see by direct calculations that

$$\langle A(u_\varepsilon) - A(v), u_\varepsilon - v\rangle \geqslant 0.$$

Hence

$$\langle A_\varepsilon(u_\varepsilon) - A(v), u_\varepsilon - v\rangle \geqslant \langle A_\varepsilon(u_\varepsilon) - A(u_\varepsilon), u_\varepsilon - v\rangle. \tag{2.4}$$

Rewrite (2.4) as following

$$\langle A_\varepsilon(u_\varepsilon), u_\varepsilon\rangle - \langle A_\varepsilon(u_\varepsilon), v\rangle - \langle A(v), u_\varepsilon - v\rangle \geqslant \langle A_\varepsilon(u_\varepsilon) - A(u_\varepsilon), u_\varepsilon - v\rangle. \tag{2.5}$$

Multiplying (2.1) by u_ε and then integrating by part we obtain

$$\langle A_\varepsilon(u_\varepsilon), u_\varepsilon\rangle = \sum_{i=1}^{n} \int_\Omega \mu_i \left(u^\alpha_{\varepsilon x_i} + \varepsilon\right)^{p_i/\alpha} u^2_{\varepsilon x_i}\, d\mathbf{x} = \int_\Omega \mathbf{f}_\varepsilon u_\varepsilon\, d\mathbf{x} \equiv (\mathbf{f}_\varepsilon, u_\varepsilon).$$

Hence from (2.5) it follows that

$$(\mathbf{f}_\varepsilon, u_\varepsilon) - \langle A_\varepsilon(u_\varepsilon), v\rangle - \langle A(v), u_\varepsilon - v\rangle \geqslant \langle A_\varepsilon(u_\varepsilon) - A(u_\varepsilon), u_\varepsilon - v\rangle. \tag{2.6}$$

Passing to the limit when $\varepsilon \to 0$ we obtain (see Remark 5 below)

$$(\mathbf{f}, u) - \langle \chi, v\rangle - \langle A(v), u - v\rangle \geqslant 0. \tag{2.7}$$

Now multiplying (2.1) by u and integrating by parts we have

$$\langle A_\varepsilon(u_\varepsilon), u\rangle = \sum_{i=1}^{n} \int_\Omega \mu_i \left(u^\alpha_{\varepsilon x_i} + \varepsilon\right)^{p_i/\alpha} u_{\varepsilon x_i} u_{x_i}\, d\mathbf{x} = \int_\Omega \mathbf{f}_\varepsilon u\, d\mathbf{x} = (\mathbf{f}_\varepsilon, u).$$

Passing to the limit when $\varepsilon \to 0$ we obtain

$$\langle \chi, u\rangle = (\mathbf{f}, u).$$

Substituting this in (2.7) we have

$$\langle \chi, u\rangle - \langle \chi, v\rangle - \langle A(v), u - v\rangle \geqslant 0 \quad \text{or} \quad \langle \chi - A(v), u - v\rangle \geqslant 0. \tag{2.8}$$

Select $v \equiv u - \lambda w$, where λ is a positive constant and $w \in \overset{\circ}{W}{}^{1,\infty}(\Omega)$. From (2.8) we conclude that

$$\lambda\langle \chi - A(u - \lambda w), w\rangle \geqslant 0 \quad \text{or} \quad \langle \chi - A(u - \lambda w), w\rangle \geqslant 0.$$

Passing to the limit when $\lambda \to 0$ (this is possible due to the Lebesgue theorem) we obtain

$$\langle \chi - A(u), w \rangle \geqslant 0 \quad \forall w \in \overset{\circ}{W}{}^{1,\infty}(\Omega).$$

Hence the functional $\chi - A(u)$ is zero, i.e.

$$\chi = A(u).$$

Thus we can pass to the limit when $\varepsilon \to 0$ in

$$\langle A_\varepsilon(u_\varepsilon), \phi \rangle = \left(c_\varepsilon g(u_\varepsilon) + f, \phi\right) \tag{2.9}$$

to obtain

$$\langle A(u), \phi \rangle = \left(cg(u) + f, \phi\right). \tag{2.10}$$

Obviously (2.9) and (2.10) are equivalent to

$$\int_\Omega \sum_{i=1}^n \mu_i \left(u_{\varepsilon x_i}^\alpha + \varepsilon\right)^{p_i/\alpha} u_{\varepsilon x_i} \phi_{x_i}\, d\mathbf{x} = \int_\Omega \left(c_\varepsilon(\mathbf{x}) g(u_\varepsilon) + f_\varepsilon(\mathbf{x})\right)\phi\, d\mathbf{x}$$

and

$$\int_\Omega \sum_{i=1}^n \mu_i |u_{x_i}|^{p_i} u_{x_i} \phi_{x_i}\, d\mathbf{x} = \int_\Omega \left(c(\mathbf{x}) g(u) + f(\mathbf{x})\right)\phi\, d\mathbf{x},$$

respectively.

The existence is proved.

Let us pass to the uniqueness. Suppose that there exist two solutions u_1 and u_2. We have

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle = \left(c(\mathbf{x})g(u_1) - c(\mathbf{x})g(u_2), u_1 - u_2\right). \tag{2.11}$$

The left-hand side of (2.11) is nonnegative, hence

$$\int_\Omega c(\mathbf{x})\left(g(u_1) - g(u_2)\right)(u_1 - u_2)\, d\mathbf{x} \geqslant 0.$$

Due to the assumptions of Theorem 1 concerning the uniqueness, the last inequality takes place only if $u_1 - u_2 \equiv 0$.

Theorem 1 is proved. □

Remark 5. When passing to the limit in (2.6) we use the fact that $\mu_i (u_{\varepsilon x_i}^\alpha + \varepsilon)^{p_i/\alpha} u_{\varepsilon x_i}$ and $\mu_i (u_{\varepsilon x_i}^\alpha)^{p_i/\alpha} u_{\varepsilon x_i} = \mu_i |u_{\varepsilon x_i}|^{p_i} u_{\varepsilon x_i}$ have the same weak limit. In fact, suppose that

$$\mu_i \left(u_{\varepsilon x_i}^\alpha + \varepsilon\right)^{p_i/\alpha} u_{\varepsilon x_i} \to \chi_1 \quad * \text{weakly in } L_\infty,$$
$$\mu_i |u_{\varepsilon x_i}|^{p_i} u_{\varepsilon x_i} \to \chi_2 \quad * \text{weakly in } L_\infty.$$

Let $|u_{\varepsilon x_i}| \leqslant C_i$. Define the function $f_i(\xi,\eta) \equiv (\xi^\alpha + |\eta|)^{p_i/\alpha}\xi$

$$f_i : [-C_i, C_i] \times [-1,1] \to \mathbf{R}.$$

For any $\phi \in L_1$, we have

$$\left|\int_\Omega \big(f_i(u_{\varepsilon x_i},\varepsilon) - f_i(u_{\varepsilon x_i},0)\big)\phi\, d\mathbf{x}\right| \to \left|\int_\Omega (\chi_1 - \chi_2)\phi\, d\mathbf{x}\right| \quad \text{when } \varepsilon \to 0.$$

Since the function f_i is continuous, for any $\sigma > 0$ there exists $\delta(\sigma)$ such that $|f_i(\xi,\eta) - f_i(\xi,0)| \leqslant \sigma$ for all $\xi \in [-C_i, C_i]$, whenever $|\eta| \leqslant \delta$. Therefore, for arbitrary $\sigma > 0$ and for $\varepsilon \leqslant \delta(\sigma)$ we obtain

$$\left|\int_\Omega \big(f_i(u_{\varepsilon x_i},\varepsilon) - f_i(u_{\varepsilon x_i},0)\big)\phi\, d\mathbf{x}\right| \leqslant \sigma \int_\Omega |\phi|\, d\mathbf{x}.$$

Thus

$$\left|\int_\Omega (\chi_1 - \chi_2)\phi\, d\mathbf{x}\right| = \lim_{\varepsilon\to 0}\left|\int_\Omega \big(f_i(u_{\varepsilon x_i},\varepsilon) - f_i(u_{\varepsilon x_i},0)\big)\phi\, d\mathbf{x}\right| \leqslant \sigma \int_\Omega |\phi|\, d\mathbf{x}.$$

Hence, $\chi_1 = \chi_2$.

Proof of Theorem 2. The existence follows from Lemma 4. In fact, from Lemma 4 we have that $\Phi(\mathbf{x}) \equiv c(\mathbf{x})g(u) + f(\mathbf{x})$ is a bounded function. This implies the a priori estimate of the solution in $C^{1+\beta}$ norm (for some $\beta \in (0,1)$) depending only on Φ_0 and n (see, for example, [5, Section 3.4]). The a priori estimate in $C^{1+\beta}$ norm implies the existence of the required solution (see, for example, [5]).

The uniqueness can be proved by standard arguments based on the maximum principle. □

Acknowledgment

We would like to thank Professor V.N. Starovoitov for the very helpful discussion on the monotonic operators and weak convergence.

References

[1] M. Clapp, M. del Pino, M. Musso, Multiple solutions for a non-homogeneous elliptic equation at the critical exponent, Proc. Roy. Soc. Edinburgh Sect. A 134 (1) (2004) 69–87.

[2] Q. Dai, J. Yang, Positive solutions of inhomogeneous elliptic equations with indefinite data, Nonlinear Anal. 58 (5–6) (2004) 571–589.

[3] I. Fragala, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (5) (2004) 715–734.

[4] H. Fujita, On the nonlinear equations $\Delta u + e^u = 0$ and $\partial v/\partial t = \Delta v + e^v$, Bull. Amer. Math. Soc. 75 (1969) 132–135.

[5] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer, 1983.

[6] S.N. Kruzhkov, Quasilinear parabolic equations and systems with two independent variables, Tr. Semin. im. I.G. Petrovskogo 5 (1979) 217–272 (in Russian); English transl. in: Topics in Modern Math., Consultant Bureau, New York, 1985.

[7] G.M. Lieberman, Gradient estimates for anisotropic elliptic equations, Adv. Differential Equations 10 (7) (2005) 767–812.
[8] J.L. Lions, Quelques methodes de resolution des problemes aux limites non lineaires, Dunod/Gauthier–Villars, Paris, 1969, 554 pp.
[9] S. Pokhozhaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Sov. Math. Dokl. 6 (1965) 1408–1411.
[10] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (3) (1992) 281–304.
[11] Al.S. Tersenov, On quasilinear non-uniformly elliptic equations in some non-convex domains, Comm. Partial Differential Equations 23 (11–12) (1998) 2165–2185.
[12] Al.S. Tersenov, Ar.S. Tersenov, On the Bernstein–Nagumo's condition in the theory of nonlinear parabolic equations, J. Reine Angew. Math. 572 (2004) 197–217.
[13] Al.S. Tersenov, Dirichlet problem for a class of quasilinear elliptic equations, Math. Notes 76 (4) (2004) 546–557.