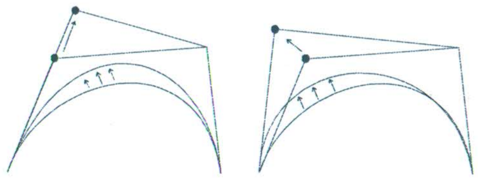

Figure 6.5 Adjustment of a control point so that the starting direction of the curve is (a) unchanged, and (b) changed

6.4 The General Bézier Curve

Given $n+1$ control points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ the Bézier curve of degree n is defined to be

$$\mathbf{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t) , \tag{6.3}$$

where

$$B_{i,n}(t) = \begin{cases} \frac{n!}{(n-i)!i!}(1-t)^{n-i}t^i, & \text{if } 0 \le i \le n \\ 0, & \text{otherwise} \end{cases} \tag{6.4}$$

are called the *Bernstein polynomials* or *Bernstein basis functions* of degree n. To distinguish Bézier curves from "rational" Bézier curves which will be introduced in Section 7.5, they are often referred to as *integral* Bézier curves. The original application of Bernstein polynomials is explored in Exercise 6.17. The polygon formed by joining the control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$ in the specified order is called the Bézier *control polygon*. It is a straightforward exercise to show that the cases $n = 1$, $n = 2$, and $n = 3$ correspond to the linear, quadratic, and cubic Bézier curves encountered in the previous sections. The Bernstein polynomials of degrees 2 and 3 are illustrated in Figure 6.6.

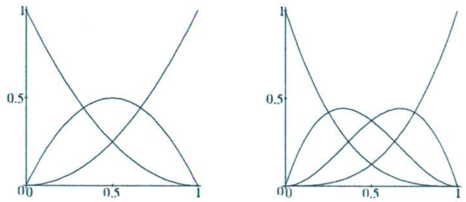

Figure 6.6 Bernstein polynomials of (a) degree 2, and (b) degree 3

The quantities $\frac{n!}{(n-i)!i!}$ are called *binomial coefficients* and are denoted by $\binom{n}{i}$ or ${}^nC_i$. Recall the convention that $0! = 1$, and therefore $\binom{n}{0} = \frac{n!}{n!0!} = 1$ and $\binom{n}{n} = \frac{n!}{0!n!} = 1$.

Example 6.3

For a Bézier cubic $n = 3$, and $B_{0,3}(t) = (1-t)^3$, $B_{1,3}(t) = 3(1-t)^2t$, $B_{2,3}(t) = 3(1-t)t^2$, and $B_{3,3}(t) = t^3$.

Example 6.4

The Bernstein polynomials of degree 4 are $B_{0,4}(t) = (1-t)^4$, $B_{1,4}(t) = 4(1-t)^3t$, $B_{2,4}(t) = 6(1-t)^2t^2$, $B_{3,4}(t) = 4(1-t)t^3$, and $B_{4,4}(t) = t^4$, as illustrated in Figure 6.7.

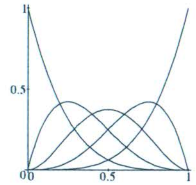

Figure 6.7 Bernstein polynomials of degree 4

The binomial coefficients arise in the result known as the *binomial theorem*.

Theorem 6.5 (Binomial)

For any natural number n, and any real numbers x and y

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i .$$

Example 6.6

Expand $(x+y)^3$ using the binomial theorem. Then

$$\begin{aligned} (x+y)^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3 . \end{aligned}$$

Example 6.7

To expand $((1-t)+t)^3$ using the binomial theorem, let $x = 1-t$ and $y = t$, and apply the result obtained in Example 6.6

$$((1-t)+t)^3 = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 .$$

It follows that $B_{0,3}(t) + B_{1,3}(t) + B_{2,3}(t) + B_{3,3}(t) = 1$.

EXERCISES

6.13. By expanding $((1-t)+t)^4$, show that the Bernstein basis functions of degree 4 sum to 1.

6.14. Determine the Bernstein polynomials of degree 5.

6.15. Show that $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$.

6.16. Show that $\int_0^1 B_{i,3}(t)\, dt = \frac{1}{4}$, for $i = 0, \ldots, 3$.

6.17. Bernstein polynomials first appeared in a proof of the Weierstrass theorem which states that any continuous function can be approximated by a polynomial function to within any specified tolerance. The Bernstein approximation $B(t)$ of degree n of a function $f(t)$ over an interval $[0, 1]$ is defined to be

$$B(t) = \sum_{i=0}^{n} f(t_i) B_{i,n}(t) ,$$

where $t_i = \frac{i}{n}$. The proof of the theorem states that for any tolerance ε there is a choice of n for which

$$|f(t) - B(t)| < \varepsilon ,$$

that is, the approximation deviates from the actual function by less than the tolerance ε. The main limitation of the approximation is that for a given ε, the choice of n is not easily determined.

(a) Plot the Bernstein approximations of degree 5, 9, and 13 for the function $f(t) = \sin(\pi t)$ over the interval $[0, 1]$.

(b) For each approximation, plot the error function

$$\text{err}(t) = |\sin(\pi t) - B(t)| ,$$

and hence determine the maximum absolute error of the approximations.

(c) Make a guess at the value of n for which the Bernstein approximation has error less than 0.01 over the interval $[0, 1]$.

6.5 Properties of the Bernstein Polynomials

The Bernstein polynomials have a number of important properties which give rise to properties of Bézier curves.

Partition of Unity: The Bernstein polynomials of degree n sum to one

$$\sum_{i=0}^{n} B_{i,n}(t) = 1, \quad t \in [0, 1] .$$

Positivity: The Bernstein polynomials are non-negative on the interval $[0, 1]$,

$$B_{i,n}(t) \geq 0, \quad t \in [0, 1] .$$

Symmetry:

$$B_{n-i,n}(t) = B_{i,n}(1-t) , \text{ for } i = 0, \ldots, n .$$

Therefore, the graph of $B_{n-i,n}(t)$ is a reflection of the graph of $B_{i,n}(1-t)$. This can be observed in Figures 6.6 and 6.7 which show plots of the quadratic, cubic, and quartic Bernstein polynomials.

Recursion: The Bernstein polynomials of degree n can be expressed in terms of the polynomials of degree $n-1$

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t) ,$$

for $i = 0, \ldots, n$, where $B_{-1,n-1}(t) = 0$ and $B_{n,n-1}(t) = 0$.

The partition of unity and positivity properties give rise to two important properties of Bézier curves, namely, invariance under transformations and the convex hull property. These properties are derived in Section 6.7. As a consequence of the symmetry property, a symmetrical control polygon gives rise to a symmetrical curve. The recursion property gives rise to the de Casteljau algorithm described in Section 6.8.

Proof

(Partition of unity) Applying the binomial theorem to $((1-t)+t)^n = 1$ gives

$$((1-t)+t)^n = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i = \sum_{i=0}^{n} B_{i,n}(t) = 1 .$$

(Recursion) The recursion property is proved as follows. By definition,

$$\begin{aligned} B_{i,n-1}(t) &= \binom{n-1}{i}(1-t)^{n-1-i}t^i \text{ , and} \\ B_{i-1,n-1}(t) &= \binom{n-1}{i-1}(1-t)^{n-i}t^{i-1} . \end{aligned}$$

For $i = 0$,

$$B_{0,n}(t) = (1-t)^n = (1-t)B_{0,n-1}(t) + tB_{-1,n-1}(t)$$

since $B_{-1,n-1}(t) = 0$. Similarly, for $i = n$,

$$B_{n,n}(t) = t^n = (1-t)B_{n,n-1}(t) + tB_{n-1,n-1}(t)$$

since $B_{n,n-1}(t) = 0$. For $1 \le i \le n-1$,

$$\begin{aligned} (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t) &= \binom{n-1}{i}(1-t)^{n-i}t^i + \binom{n-1}{i-1}(1-t)^{n-i}t^i \\ &= \left(\binom{n-1}{i} + \binom{n-1}{i-1}\right)(1-t)^{n-i}t^i . \end{aligned}$$

Applying Exercise 6.15,

$$(1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t) = \binom{n}{i}(1-t)^{n-i}t^i = B_{i,n}(t) .$$

□

The proofs of the properties of positivity and symmetry are left as exercises.

EXERCISES

6.18. Prove the positivity property.

6.19. Prove the symmetry property.

6.20. Show that $\sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) = t$. Deduce the *linear precision property* that if $\mathbf{b}_i = \left(1 - \frac{i}{n}\right)\mathbf{a} + \frac{i}{n}\mathbf{b}$ for some fixed points $\mathbf{a}$ and $\mathbf{b}$ (so the control points are evenly distributed along the line segment $\overline{\mathbf{ab}}$), then the resulting Bézier curve $\mathbf{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t)$ is the straight line segment $\overline{\mathbf{ab}}$.

6.21. Let $\mathbf{B}(t)$ be a Bézier curve of degree n with control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$. Let $\mathbf{C}(t)$ be the Bézier curve of degree $n+1$ with control points $\mathbf{c}_0 = \mathbf{b}_0$, $\mathbf{c}_{n+1} = \mathbf{b}_n$, and $\mathbf{c}_i = (1-\alpha_i)\mathbf{b}_i + \alpha_i \mathbf{b}_{i-1}$ where $\alpha_i = i/(n+1)$, for $i = 1, \ldots, n$. Show that $\mathbf{C}(t) = \mathbf{B}(t)$ for all $t \in [0,1]$. The process of representing a Bézier curve of degree n by a Bézier curve of higher degree is called *degree raising*. Degree-raising algorithms are used to increase the number of control points to give greater freedom for designing curve shapes.

6.6 Convex Hulls

An important and useful property of Bézier curves is that of the convex hull property (CHP) which will be derived in Section 6.7. The CHP and the de Casteljau algorithm, derived in Section 6.8, lead naturally to geometric algorithms for rendering a Bézier curve, and for finding the points of intersection of two Bézier curves. In order to describe the CHP it is necessary to define the convex hull of a set of points. Given a set of points $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n\}$ the *convex hull* of X, denoted by $\mathrm{CH}\{X\}$, is defined to be the set of points

$$\mathrm{CH}\{X\} = \left\{ a_0\mathbf{x}_0 + \cdots + a_n\mathbf{x}_n \;\middle|\; \sum_{i=0}^{n} a_i = 1 \,, a_i \ge 0 \right\} . \tag{6.5}$$

For points in a plane, the convex hull $\mathrm{CH}\{X\}$ may be visualized as follows. Imagine an "elastic band" placed around the entire set of points. The band is permitted to shrink around the points to form a polygon, the vertices of which are a subset of the original set of points. The region bounded by the polygon is the convex hull of the set of points.

The definition of the convex hull is valid for points in space. The intuitive elastic band is replaced by an "elastic balloon" which is permitted to shrink around the points to form a polyhedron. The convex hull is the region bounded by the polyhedron. Several examples of convex hulls are illustrated in Figure 6.8.

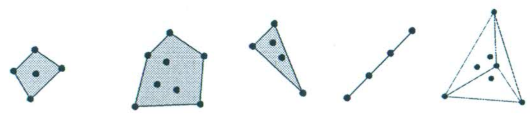

Figure 6.8

6.7 Properties of Bézier Curves

Theorem 6.8

A Bézier curve $\mathbf{B}(t)$ of degree n with control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$ satisfies the following properties.

Endpoint Interpolation Property: $\mathbf{B}(0) = \mathbf{b}_0$ and $\mathbf{B}(1) = \mathbf{b}_n$.

Endpoint Tangent Property:

$$\mathbf{B}'(0) = n\left(\mathbf{b}_1 - \mathbf{b}_0\right) \quad \text{and} \quad \mathbf{B}'(1) = n\left(\mathbf{b}_n - \mathbf{b}_{n-1}\right).$$

Convex Hull Property (CHP): For all $t \in [0,1]$, $\mathbf{B}(t) \in \mathrm{CH}\{\mathbf{b}_0, \ldots, \mathbf{b}_n\}$. Thus *every point of a Bézier curve lies inside the convex hull of its defining control points*. The convex hull of the control points is often referred to as the convex hull of the Bézier curve.

Invariance under Affine Transformations: Let T be an (affine) transformation (for example, a rotation, reflection, translation, or scaling). Then

$$\mathsf{T}\left(\sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t)\right) = \sum_{i=0}^{n} \mathsf{T}\left(\mathbf{b}_i\right) B_{i,n}(t).$$

Variation Diminishing Property (VDP): For a planar Bézier curve $\mathbf{B}(t)$, the VDP states that the number of intersections of a given line with $\mathbf{B}(t)$ is less than or equal to the number of intersections of that line with the control polygon.

Proof

The proof of the endpoint interpolation property is Exercise 6.23. The endpoint tangent property follows from Theorem 7.3 which will be proved later.

(Convex Hull Property) From the definition of the convex hull expressed in Equation (6.5) it is sufficient to show that every point $\mathbf{B}(t)$ on a Bézier curve has the form $a_0\mathbf{b}_0 + \cdots + a_n\mathbf{b}_n$ for some constants a_i satisfying $\sum_{i=0}^{n} a_i = 1$. Let $a_i = B_{i,n}(t)$, then positivity implies $a_i \geq 0$, the partition of unity implies that $\sum_{i=0}^{n} a_i = 1$, and the proof is complete. Figure 6.9 illustrates the CHP for a cubic Bézier curve.

(Affine Invariance) Let an affine transformation T be given by

$$(x', y') = (ax + by + c, dx + ey + f),$$

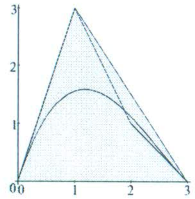

Figure 6.9 Convex hull property for a cubic Bézier curve

and let a Bézier curve of degree n have control points $\mathbf{b}_i(p_i, q_i)$ for $i = 0, \ldots, n$. Then

$$\mathbf{B}(t) = (x(t), y(t)) = \left(\sum_{i=0}^{n} p_i B_{i,n}(t), \sum_{i=0}^{n} q_i B_{i,n}(t)\right).$$

Applying the transformation yields

$$\begin{aligned} \mathsf{T}(\mathbf{B}(t)) = \Bigg(& a\sum_{i=0}^{n} p_i B_{i,n}(t) + b\sum_{i=0}^{n} q_i B_{i,n}(t) + c, \\ & d\sum_{i=0}^{n} p_i B_{i,n}(t) + e\sum_{i=0}^{n} q_i B_{i,n}(t) + f\Bigg). \end{aligned}$$

Then, by partition of unity, $\sum_{i=0}^{n} B_{i,n}(t) = 1$, and

$$\begin{aligned} \mathsf{T}(\mathbf{B}(t)) &= \Bigg(a\sum_{i=0}^{n} p_i B_{i,n}(t) + b\sum_{i=0}^{n} q_i B_{i,n}(t) + c\sum_{i=0}^{n} B_{i,n}(t), \\ &\qquad d\sum_{i=0}^{n} p_i B_{i,n}(t) + e\sum_{i=0}^{n} q_i B_{i,n}(t) + f\sum_{i=0}^{n} B_{i,n}(t)\Bigg) \\ &= \left(\sum_{i=0}^{n}(ap_i + bq_i + c)B_{i,n}(t), \sum_{i=0}^{n}(dp_i + eq_i + f)B_{i,n}(t)\right) \\ &= \sum_{i=0}^{n}(ap_i + bq_i + c, dp_i + eq_i + f)B_{i,n}(t) \\ &= \sum_{i=0}^{n} \mathsf{T}(\mathbf{b}_i)B_{i,n}(t). \end{aligned}$$

□

Example 6.9

Consider a cubic Bézier curve with vertices $\mathbf{b}_0(1,0)$, $\mathbf{b}_1(2,3)$, $\mathbf{b}_2(5,4)$, and $\mathbf{b}_3(2,1)$. To apply a rotation through an angle $\pi/4$ about the origin in the anticlockwise direction to the curve, it is sufficient to apply the rotation matrix $\mathsf{Rot}(\pi/4)$ to the homogeneous coordinates of the control points:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 5 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \cos\pi/4 & \sin\pi/4 & 0 \\ -\sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.707 & 0.707 & 1.0 \\ -0.707 & 3.536 & 1.0 \\ 0.707 & 6.364 & 1.0 \\ 0.707 & 2.121 & 1.0 \end{pmatrix}.$$

The control points of the rotated curve are $\mathbf{b}_0(0.707, 0.707)$, $\mathbf{b}_1(-0.707, 3.536)$, $\mathbf{b}_2(0.707, 6.364)$, and $\mathbf{b}_3(0.707, 2.121)$. The curve and its rotated image are illustrated in Figure 6.10.

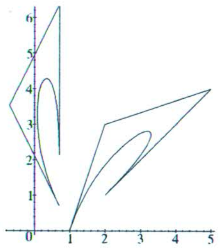

Figure 6.10 Application of a rotation to a cubic Bézier curve

Figure 6.11 illustrates two lines intersecting a Bézier curve and its control polygon. The upper line intersects the polygon in two points but does not intersect the curve. The lower line intersects both the polygon and the curve in two points. In both cases, the number of intersections with the given line is equal to or greater than the number of intersections of the line with the curve. Thus the variation diminishing property is satisfied. The proof of the variation diminishing property is beyond the scope of this book, and the reader is referred to [15].

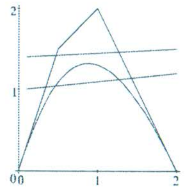

Figure 6.11 Variation diminishing property

EXERCISES

6.22. Plot the cubic Bézier curve defined by control points $\mathbf{b}_0(0,1)$, $\mathbf{b}_1(2,5)$, $\mathbf{b}_2(4,6)$, and $\mathbf{b}_3(8,1)$. On the same plot, draw the control polygon. Observe that the resulting curve satisfies the convex hull property. Next plot the Bézier cubic with control points $\mathbf{b}_0(1,1)$, $\mathbf{b}_1(3.4,1.8)$, $\mathbf{b}_2(6,6.5)$, and $\mathbf{b}_3(9,1)$. Does the newly displayed curve violate the convex hull property? Explain.

6.23. Prove the endpoint interpolation property for the general Bézier curve: $\mathbf{B}(0) = \mathbf{b}_0$ and $\mathbf{B}(1) = \mathbf{b}_n$.

6.24. Prove that when the control points are collinear, the resulting Bézier curve is a straight line segment.

6.25. Determine the control points of the image of the Bézier curve with control points $\mathbf{b}_0(0,0)$, $\mathbf{b}_1(2,1)$, $\mathbf{b}_2(3,-1)$, and $\mathbf{b}_3(1,-2)$ when the following transformations have been applied

(a) a translation of 3 units in the x-direction and 4 units in the y-direction,

(b) a rotation about the origin through an angle of $\pi/2$ radians in an anti-clockwise direction,

(c) a reflection in the line $y = x$.

For each transformation plot the image curve and its control polygon.

6.26. The basis functions $B_{0,3}(t) = (1-t)^2$, $B_{1,3}(t) = 2t(1-t)^2$, $B_{2,3}(t) = 2t^2(1-t)$, $B_{0,3}(t) = t^2$ give rise to a representation for cubic curves $\mathbf{B}(t) = \sum_{i=0}^{3} \mathbf{b}_i B_{i,3}(t)$.

(a) Show that if $\mathbf{b}_1 = \mathbf{b}_2$ then $\mathbf{B}(t)$ is a quadratic curve with control polygon $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{b}_3$.

(b) Show that the representation satisfies end interpolation and tangent properties similar to Bézier curves.

6.8 The de Casteljau Algorithm

The de Casteljau algorithm provides a method for evaluating the point on a Bézier curve corresponding to the parameter value $t \in [0, 1]$. In Section 6.9 it will be shown that the same algorithm can be used to divide a curve into two curve segments. For the case of a cubic Bézier curve with control points $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, and for a specified parameter value $t \in [0, 1]$, the de Casteljau algorithm is expressed by the recursive formula

$$\begin{cases} \mathbf{b}_i^0 = \mathbf{b}_i, \\ \mathbf{b}_i^j = (1-t)\mathbf{b}_i^{j-1} + t\mathbf{b}_{i+1}^{j-1}, \end{cases}$$

for $j = 1, 2, 3$ and $i = 0, \ldots, 3 - j$. The formula generates a triangular set of values (6.6) for which $\mathbf{b}_0^3 = \mathbf{B}(t)$ for the specified value of t:

$$\begin{array}{cccc} \mathbf{b}_0^0 & \mathbf{b}_1^0 & \mathbf{b}_2^0 & \mathbf{b}_3^0 \\ \mathbf{b}_0^1 & \mathbf{b}_1^1 & \mathbf{b}_2^1 & \\ \mathbf{b}_0^2 & \mathbf{b}_1^2 & & \\ \mathbf{b}_0^3 & & & \end{array} \qquad (6.6)$$

Example 6.10

A cubic Bézier curve has control points $\mathbf{b}_0(1.0, 1.0)$, $\mathbf{b}_1(2.0, 7.0)$, $\mathbf{b}_2(8.0, 6.0)$, and $\mathbf{b}_3(12.0, 2.0)$. The point $\mathbf{B}(0.25)$ is determined by applying the de Casteljau algorithm with $t = 0.25$. Then

$$\begin{aligned} \mathbf{b}_0^1 &= \tfrac{3}{4}(1.0, 1.0) + \tfrac{1}{4}(2.0, 7.0) = (1.25, 2.5), \\ \mathbf{b}_1^1 &= \tfrac{3}{4}(2.0, 7.0) + \tfrac{1}{4}(8.0, 6.0) = (3.5, 6.75), \\ \mathbf{b}_2^1 &= \tfrac{3}{4}(8.0, 6.0) + \tfrac{1}{4}(12.0, 2.0) = (9.0, 5.0), \\ \mathbf{b}_0^2 &= \tfrac{3}{4}(1.25, 2.5) + \tfrac{1}{4}(3.5, 6.75) = (1.8125, 3.5625), \text{ etc.} \end{aligned}$$

The algorithm gives the following table of points:

$$\begin{array}{ccccccc} (1.0, 1.0) & & (2.0, 7.0) & & (8.0, 6.0) & & (12.0, 2.0) \\ & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & \\ (1.25, 2.5) & & (3.5, 6.75) & & (9.0, 5.0) & & \\ & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & & & \\ (1.8125, 3.5625) & & (4.875, 6.3125) & & & & \\ & \tfrac{3}{4}\downarrow\swarrow\tfrac{1}{4} & & & & & \\ (2.578, 4.25) & & & & & & \end{array}$$

The algorithm yields $\mathbf{B}(0.25) = (2.578, 4.25)$. Geometrically, each step of the algorithm is a linear interpolation of the control polygon as illustrated in Figure 6.12.

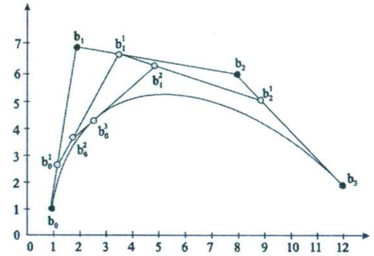

Figure 6.12 The de Casteljau algorithm with $t = 0.25$

Theorem 6.11

Let a Bézier curve of degree n be given by control points $\mathbf{b}_0, \ldots, \mathbf{b}_n$, and let $t \in [0, 1]$ be any parameter value. Then $\mathbf{B}(t) = \mathbf{b}_0^n$, where $\mathbf{b}_i^0 = \mathbf{b}_i$, and

$$\mathbf{b}_i^j = \mathbf{b}_i^{j-1}(1-t) + \mathbf{b}_{i+1}^{j-1}t,$$

for $j = 1, \ldots, n$, and $i = 0, \ldots, n - j$.

Proof

The de Casteljau algorithm follows from the recursion property of the Bernstein polynomials

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t). \qquad (6.7)$$

Then

$$\begin{aligned}\mathbf{B}(t) &= \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t) = \sum_{i=0}^{n} \mathbf{b}_i \left((1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)\right) \\ &= \sum_{i=0}^{n} \mathbf{b}_i (1-t) B_{i,n-1}(t) + \sum_{i=0}^{n} \mathbf{b}_i t B_{i-1,n-1}(t) \,.\end{aligned}$$

Since $B_{n,n-1}(t) = 0$, and $B_{-1,n-1}(t) = 0$ it follows that

$$\mathbf{B}(t) = \sum_{i=0}^{n-1} \mathbf{b}_i (1-t) B_{i,n-1}(t) + \sum_{i=1}^{n} \mathbf{b}_i t B_{i-1,n-1}(t) \,.$$

Next renumber the second summation by replacing i by $i+1$,

$$\begin{aligned}\mathbf{B}(t) &= \sum_{i=0}^{n-1} \mathbf{b}_i (1-t) B_{i,n-1}(t) + \sum_{i=0}^{n-1} \mathbf{b}_{i+1} t B_{i,n-1}(t) \\ &= \sum_{i=0}^{n-1} \left(\mathbf{b}_i (1-t) + \mathbf{b}_{i+1} t\right) B_{i,n-1}(t) \,.\end{aligned}$$

Set $\mathbf{b}_i^1 = \mathbf{b}_i(1-t) + \mathbf{b}_{i+1}t = \mathbf{b}_i^0(1-t) + \mathbf{b}_{i+1}^0 t$ for $i = 0, \ldots, n-1$, then

$$\mathbf{B}(t) = \sum_{i=0}^{n-1} \mathbf{b}_i^1 B_{i,n-1}(t) \,. \qquad (6.8)$$

Equation (6.8) expresses $\mathbf{B}(t)$ as a Bézier curve of degree $n-1$ with control points $\mathbf{b}_0^1, \ldots, \mathbf{b}_{n-1}^1$. Applying a similar argument yields

$$\mathbf{B}(t) = \sum_{i=0}^{n-2} \mathbf{b}_i^2 B_{i,n-2}(t) \,,$$

where $\mathbf{b}_{i+1}^2 = \mathbf{b}_i^1(1-t) + \mathbf{b}_{i+1}^1 t$ for $i = 0, \ldots, n-2$. In general,

$$\mathbf{B}(t) = \sum_{i=0}^{n-j} \mathbf{b}_i^j B_{i,n-j}(t) \,,$$

where $\mathbf{b}_i^j = \mathbf{b}_i^{j-1}(1-t) + \mathbf{b}_{i+1}^{j-1} t$ for $i = 0, \ldots, n-j$. In particular, $j = n$ gives

$$\mathbf{B}(t) = \sum_{i=0}^{0} \mathbf{b}_i^n B_{i,n-n}(t) = \mathbf{b}_0^n \,.$$

□

EXERCISES

6.27. A cubic Bézier curve has control points $\mathbf{b}_0(1,0)$, $\mathbf{b}_1(3,3)$, $\mathbf{b}_2(5,5)$, and $\mathbf{b}_3(7,2)$. Evaluate the point $\mathbf{B}(0.25)$ by (a) applying the de Casteljau algorithm, and (b) substituting $t = 0.25$ into the defining equation of the Bézier curve. Make a sketch illustrating the points derived in applying de Casteljau algorithm.

6.28. Apply the de Casteljau algorithm to the quartic Bézier curve with control points $\mathbf{b}_0(3.0, 3.0)$, $\mathbf{b}_1(4.0, 2.0)$, $\mathbf{b}_2(-1.0, 0.0)$, $\mathbf{b}_3(6.0, 1.0)$, and $\mathbf{b}_4(8.0, 5.0)$, and evaluate the point $\mathbf{B}(0.6)$.

6.29. (Used in Theorem 6.13) Prove that the intermediate control points defined in the de Casteljau algorithm satisfy

$$\mathbf{b}_k^j = \sum_{i=0}^{j} B_{i,j}(t)\mathbf{b}_{i+k} .$$

6.30. Show that

$$\begin{aligned}(1-t)B_i^n(t) &= \left(\frac{n+1-i}{n+1}\right) B_i^{n+1}(t) \,, \\ tB_i^n(t) &= \left(\frac{i+1}{n+1}\right) B_{i+1}^{n+1}(t) \,.\end{aligned}$$

6.31. (Used in Theorem 6.13) Use Exercise 6.29 (or otherwise) to show that $B_{i,n}(\alpha t) = \sum_{j=0}^{n} B_{i,j}(\alpha) B_{j,n}(t)$.

6.9 Subdivision of a Bézier Curve

A Bézier curve is generally defined over the interval $[0,1]$ and given by $\mathbf{B}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_{i,n}(t)$. On occasions, only a part of a curve is of interest. For instance, suppose that a Bézier curve is "cut" at the parameter value $t = \alpha$ to give two curve segments, denoted by $\mathbf{B}_{\text{left}}(t)$ and $\mathbf{B}_{\text{right}}(t)$, defined over the intervals $[0, \alpha]$, and $[\alpha, 1]$ as shown in Figure 6.13. Since $\mathbf{B}_{\text{left}}(t)$ and $\mathbf{B}_{\text{right}}(t)$ are polynomial curves they can be represented in Bézier form over the interval $[0,1]$. Theorem 6.13 will show that to determine the control points of $\mathbf{B}_{\text{left}}(t)$ and $\mathbf{B}_{\text{right}}(t)$ it is sufficient to apply the de Casteljau algorithm to $\mathbf{B}(t)$ with $t = \alpha$. For a cubic Bézier curve, the theorem implies that the control points of $\mathbf{B}_{\text{left}}(t)$ are $\mathbf{b}_0^0, \mathbf{b}_0^1, \mathbf{b}_0^2, \mathbf{b}_0^3$, and the control points of $\mathbf{B}_{\text{right}}(t)$ are $\mathbf{b}_0^3, \mathbf{b}_1^2, \mathbf{b}_2^1, \mathbf{b}_3^0$. The two sets of points are observed to be two edges of the triangle of control