Theorem 2.7. *Let a function $f \in C^2[a,b]$ and a cubic spline $s \in S_{\Delta,3}$ be given such that they share equal values at the vertices, compare (2.5). Then*

$$\|f''\|_2^2 \; - \; \|s''\|_2^2 \; = \; \|f'' - s''\|_2^2 \tag{2.8}$$

if one of the following three conditions is satisfied:

(a) $s''(a) = s''(b) = 0;$
(b) $s'(a) = f'(a), \quad s'(b) = f'(b);$
(c) $f'(a) = f'(b), \quad s'(a) = s'(b), \quad s''(a) = s''(b).$

Proof. In each of the cases (a)-(c), the expression $([f' - s']s'')(x)\big|_{x=a}^{x=b}$ in (2.6) vanishes and the identity (2.6) becomes the identity (2.8). □

Corollary 2.8. *For given values $f_0, \ldots, f_N \in \mathbb{R}$, an interpolating cubic spline $s \in S_{\Delta,3}$ with $s''(a) = s''(b) = 0$ has the smallest curvature among all sufficiently smooth interpolating functions, i.e.,*

$$\|s''\|_2 \le \|f''\|_2$$

for each function $f \in C^2[a,b]$ with $f(x_k) = f_k$ for $k = 0, \ldots, N$.

Proof. The estimate follows directly from Theorem 2.7 for splines with property (a). □

The estimate given in Corollary 2.8 also holds for cubic splines that satisfy conditions (b) or (c) in Theorem 2.7 (after some modifications are made to the respective prerequisites).

Remark 2.9. (1) Using property (2.8), one easily shows that each of the conditions (a), (b) or (c) in Theorem 2.7 implies the uniqueness of the interpolating cubic spline (Exercise 2.2).
(2) $\|f''\|_2$ is only an approximation of the average curvature of the function f. More precisely, the curvature of f at a point x is given by $f''(x)/(1 + f'(x)^2)^{3/2}$.
(3) It is the minimality property introduced in Corollary 2.8 that explains why cubic spline functions are used for interpolation in practical applications (e.g., the construction of boat hulls or the design of train tracks). △

Figure 2.3 displays a cubic spline.

Section 2.4. The calculation of interpolating cubic spline functions

2.4.1. Prerequisites. Here, the calculation of interpolating cubic splines is addressed. Starting with the general model

$$s(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3 \tag{2.9}$$
$$\text{for} \quad x \in [x_k, x_{k+1}], \qquad k = 0, 1, \ldots, N-1,$$

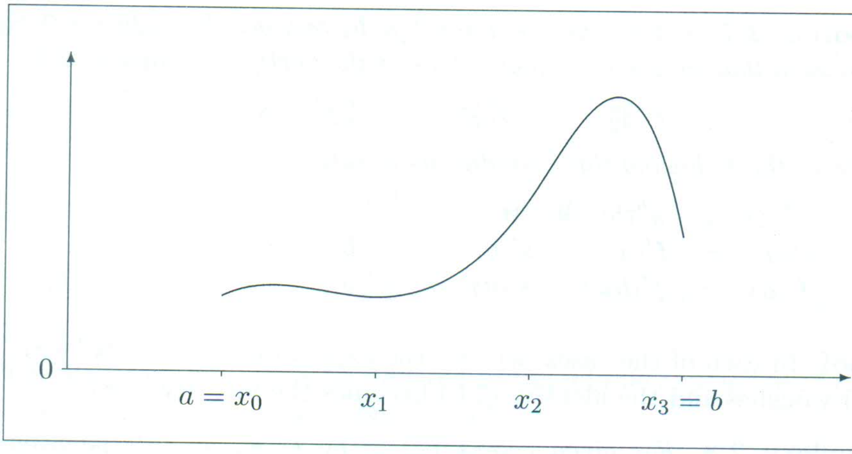

Figure 2.3. A cubic spline on $[a, b]$ for the vertices $a = x_0 < x_1 < x_2 < x_3 = b$.

for a function $s : [a, b] \to \mathbb{R}$, this section discusses how one chooses the coefficients a_k, b_k, c_k and d_k for $k = 0, 1, \dots, N-1$ such that the function s is twice continuously differentiable[1] on the interval $[a, b]$ and, additionally, interpolates the given values $f_0, \dots, f_N \in \mathbb{R}$ exactly for all vertices, i.e.,

$$s(x_k) = f_k \qquad \text{for } k = 0, \dots, N. \tag{2.10}$$

The subsequent lemma reduces the above problem to the solution of a linear system of equations that uses the notation

$$h_k := x_{k+1} - x_k, \qquad k = 0, \dots, N-1. \tag{2.11}$$

Lemma 2.10. *If $N+1$ real numbers $s''_0, \dots, s''_N \in \mathbb{R}$ satisfy the following $N-1$ coupled equations*

$$h_{k-1}s''_{k-1} + 2(h_{k-1} + h_k)s''_k + h_k s''_{k+1} = \underbrace{6\frac{f_{k+1} - f_k}{h_k} - 6\frac{f_k - f_{k-1}}{h_{k-1}}}_{=:\, g_k}, \qquad k = 1, \dots, N-1, \tag{2.12}$$

then (2.9) together with defining

$$c_k := \frac{s''_k}{2}, \qquad a_k := f_k, \qquad d_k := \frac{s''_{k+1} - s''_k}{6h_k}, \tag{2.13}$$

$$b_k := \frac{f_{k+1} - f_k}{h_k} - \frac{h_k}{6}(s''_{k+1} + 2s''_k), \tag{2.14}$$

for $k = 0, \dots, N-1$, yields a cubic spline function $s \in S_{\Delta,3}$ which satisfies the interpolation conditions (2.10).

---

[1] and therefore, in fact, is a cubic spline

Proof. Using the notation

$$p_k(x) \;=\; a_k \;+\; b_k(x-x_k) \;+\; c_k(x-x_k)^2 \;+\; d_k(x-x_k)^3 \;\in\; \Pi_3 \qquad (k \;=\; 0,1,\ldots,N-1),$$

one obtains the following identities for $k = 0,1,\ldots,N-1$:

$$\begin{aligned}
p_k(x_k) &= a_k \;=\; f_k,\\
p''_{k+1}(x_{k+1}) &= 2c_{k+1} \;=\; s''_{k+1} \;=\; s''_k \;+\; 6d_kh_k\\
&= p''_k(x_{k+1}) \qquad (k \le N-2),\\
p_k(x_{k+1}) &= a_k \;+\; b_kh_k \;+\; c_kh_k^2 \;+\; d_kh_k^3\\
&= f_k \;\text{——}\;\text{«}\;\text{——}\; \frac{s''_k}{2}h_k^2 \;+\; \frac{s''_{k+1} \;-\; s''_k}{6}h_k^2 \;\overset{(*)}{=}\; f_{k+1},
\end{aligned}$$

where the identity $(*)$ results from (2.14). One obtains the continuity of the first derivative s' from

$$\begin{aligned}
p'_{k-1}(x_k) &= b_{k-1} \;+\; 2c_{k-1}h_{k-1} \;+\; 3d_{k-1}h_{k-1}^2\\
&\overset{(**)}{=} b_k \;=\; p'_k(x_k) \qquad (k \;=\; 1,\ldots,N-1),
\end{aligned}$$

where $(**)$ follows from substituting the alternate expressions for the coefficients given in (2.13)–(2.14) and (2.12). □

Remark 2.11. (1) In the situation described in Lemma 2.10, the $N+1$ real numbers $s''_0,\ldots,s''_N \in \mathbb{R}$ are referred to as **moments**. These are equal to the second derivatives of the spline function s at the vertices x_k,

$$s''_k \;=\; s''(x_k) \qquad \text{for} \quad k \;=\; 0,\ldots,N.$$

(2) Lemma 2.10 shows that the coefficients given in (2.9) are a direct result from the $N+1$ moments $s''_0,\ldots,s''_N$. These $N+1$ moments in turn result from the $N-1$ conditions given in equations (2.12) of this lemma, thus allowing a choice for the two remaining degrees of freedom. With respect to conditions (a)–(c) in Theorem 2.7, three possibilities are discussed after setting

$$s'_0 \;:=\; s'(x_0), \qquad s'_N \;:=\; s'(x_N):$$

Natural boundary conditions : $\quad s''_0 \;=\; s''_N \;=\; 0;$

Complete boundary conditions : $\quad s'_0 \;=\; f'_0, \qquad s'_N \;=\; f'_N$ for given $f'_0,\; f'_N \in \mathbb{R}$;

Periodic boundary conditions : $\quad s'_0 \;=\; s'_N, \qquad s''_0 \;=\; s''_N.$

The term "natural boundary condition" is justified by Corollary 2.8.

(3) Division of (2.12) by $3(h_{k-1}+h_k)$ leads to the equivalent equation

$$\text{(2.15)} \qquad \frac{h_{k-1}}{3(h_{k-1}+h_k)}s''_{k-1} + \frac{2}{3}s''_k + \frac{h_k}{3(h_{k-1}+h_k)}s''_{k+1} = 2\frac{f_{k+1}-f_k}{h_k(h_{k-1}+h_k)} - 2\frac{f_k-f_{k-1}}{h_{k-1}(h_{k-1}+h_k)},$$

where the left-hand side represents an approximation to s''_k and the right-hand side represents a difference approximation to $f''(x_k)$. Additional details can be found in the proof of Lemma 2.15. △

In Sections 2.4.2-2.4.4 to follow, conditions (2.12) for the moments are combined with various boundary conditions, respectively, and given in their matrix-vector representations, respectively.

2.4.2. Natural boundary conditions. The natural boundary conditions $s''_0 = s''_N = 0$ together with (2.12) lead to the following system of equations:

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 & \dots & 0 \\ h_1 & 2(h_1+h_2) & h_2 & \ddots & \vdots \\ 0 & h_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & h_{N-2} \\ 0 & \dots & 0 & h_{N-2} & 2(h_{N-2}+h_{N-1}) \end{bmatrix} \begin{pmatrix} s''_1 \\ \vdots \\ s''_{N-1} \end{pmatrix} = \begin{pmatrix} g_1 \\ \vdots \\ g_{N-1} \end{pmatrix}.$$

2.4.3. Complete boundary conditions. The complete boundary conditions

$$f'_0 \overset{!}{=} s'_0 = b_0,$$

$$f'_N \overset{!}{=} s'_N = b_{N-1} + 2c_{N-1}h_{N-1} + 3d_{N-1}h^2_{N-1}$$

together with (2.13)-(2.14) lead to the two additional conditions

$$\text{(2.16)} \qquad 2h_0s''_0 + h_0s''_1 = -6f'_0 + 6\frac{f_1-f_0}{h_0} =: g_0,$$

$$\text{(2.17)} \qquad h_{N-1}s''_{N-1} + 2h_{N-1}s''_N = 6f'_N - 6\frac{f_N-f_{N-1}}{h_{N-1}} =: g_N.$$